Convergence to a self-similar solution for a one-phase Stefan problem arising in corrosion theory

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Steel corrosion plays a central role in different technological fields. In this paper, we consider a simple case of a corrosion phenomenon which describes a pure iron dissolution in sodium chloride. This article is devoted to prove rigorously that under rather general hypotheses on the initial data, the solution of this iron dissolution model converges to a self-similar profile as $t \to +\infty$. We will do so for an equivalent formulation as presented in the book of Avner Friedman about parabolic equations [9]. In order to prove the convergence result, we apply a comparison principle together with suitable upper and lower solutions.

Key Words: Stefan problem; free-boundary problem; self-similar solution; upper and lower solutions; large-time behavior; corrosion theory.

2020 Mathematics Subject Classification: 35R35, 35C06, 35B40 (Primary); 80A22 (Secondary)

Introduction

We consider a pure iron steel in contact with an aqueous solution of sodium chloride (NaCl). One of the major failure mechanisms in aggressive aqueous solution is pitting corrosion. It is generally associated to the presence of a special anion, namely the chloride ion. The presence of such an ion leads to the formation of small isolated holes (pits) in the surface of the steel that may reach a considerable depth. Thus, the life cycle of a stainless alloy decreases in the presence of corrosion. For many decades, several computational models have been developed to study and predict the time evolution of pitting corrosion. The objective of these models is to provide powerful means to simulate the propagation of localized corrosion, mainly pitting corrosion, in various environments, and to reduce its impact. In this paper, we focus on a physical model which aims to describe the propagation process of one individual corrosion pit. To do so, we consider the one-dimensional mathematical model given in the following two references : [16, 17]. The model describes a pit growing in stable corrosion mode which refers to the case where propagation of the pit occurs for long periods of time since critical conditions inside it prevents the stop of its propagation [17]. Such a case is possible once a salt film is formed at the bottom of the pit.

In this model, we only describe the evolution of the metal atom concentration C that arises from pit dissolution. The evolution of the chloride concentration as well as the sodium concentration are not taken into account. Figure 1 represents a one dimensional stable pit where corrosion only occurs at the bottom of the pit (the walls of the pit do not move). Note that a one-dimensional pit should be represented as an interval with moving boundary $\left[0, z_d(t)\right)$ but in order to indicate all the necessary physical parameters, we represent it as a rectangle in Figure 1.

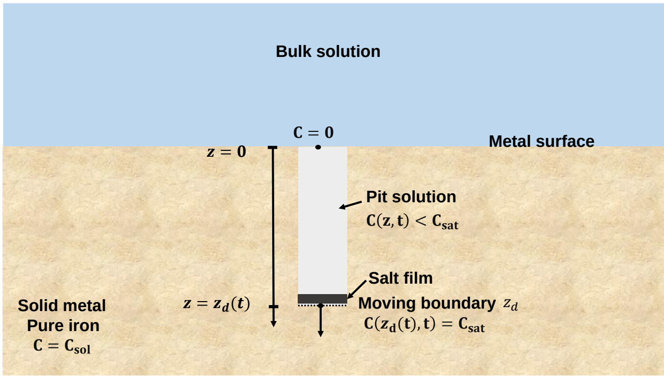


Figure 1. A one dimensional corrosion pit.

It is composed of the following domains :

(1) Solid metal with temporally and spatially constant metal atom concentration. In the following, we represent the metal atom concentration by C_{sol}.

(2) Salt film at the pit bottom : the metal ions released from the solid metal combine with chlorides in the solution, forming metal chloride salt at the bottom of the pit. The more ions are released, the more salt is formed, until the saturation concentration of the salt is reached [17]. Thus, at the bottom of the pit, the iron ion concentration remains constant and equal to the saturation limit C_{sat}.

(3) Pit solution with temporally and spatially varying concentration of dissolved iron ions. Their concentration is laying below the saturation concentration: $C(z,t) < C_{\text{sat}}$.

(4) Bulk solution outside of the pit, characterized by the vanishing iron ion concentration $C = 0$.

The depth of the pit is given by $z_d(t)$ which indicates the position of the moving boundary at time $t > 0$ for suitable initial conditions for z_d and C.

1 The physical model

The physical model is given by

$$\begin{cases} C_t = D\, C_{zz}, & t > 0,\ 0 < z < z_d(t), \\ C(0,t) = 0, & t > 0, \\ C(z_d(t)^-, t) = C_{\text{sat}}, & t > 0, \\ \dfrac{dz_d(t)}{dt} = \dfrac{D}{C_{\text{sol}} - C_{\text{sat}}}\, C_z(z_d(t)^-, t), & t > 0, \end{cases} \tag{1.1}$$

where $D > 0$ is the diffusion coefficient of the iron ions, $z_d(t)^-$ refers to the liquid side of the moving boundary and $C_{\text{sol}} > C_{\text{sat}}$.

In the following paragraph, we focus our study on the influence of two input parameters: the diffusion coefficient D and the saturated concentration C_{sat} in Problem (1.1). These two parameters play an important role to understand the evolution of the pit depth as a function of time.

Choice of the input parameters :
In view of literature, we have found some possible values of D and C_{sat}. Indeed, it was mentioned that in the case of a one-dimensional pit, a reasonable approximation of the value of D is in the range $[7.10^{-6}, 10^{-5}]$ $(\text{cm}^2.\text{s}^{-1})$ [18] while the value of C_{sat} can be equal to 5.02 mol/L [18] or 4.2 mol/L [19]. In [17], these parameters were set to $0.85 \cdot 10^{-5}\ \text{cm}^2.\text{s}^{-1}$ for the diffusion coefficient and to 5.1 mol/L for the saturated concentration value. Thus, we will focus on these values to describe the evolution of the physics of the corrosion phenomenon. On the other hand, the value of the metal concentration C_{sol} will be set to 143 mol/L [17].

Effect of the diffusion coefficient on the evolution of the pit depth :
In order to study the influence of the diffusion coefficient D on the propagation velocity, we perform several computations as a function of D for the following choice of the physical parameters

$$C_{\text{sat}} = 5.1\,\text{mol/L}, \quad C_{\text{sol}} = 143\,\text{mol/L} \tag{1.2}$$

with the initial values given by (see Figure 2) :

- Initial pit depth : $z_d^0 = 1\mu\text{m}$.
- At the pit entrance $(z = 0)$: $C(0,0) = 10^{-6}$ mol/L.
- At the pit bottom $\big(z = z_d(0)\big)$: $C(z_d(0), 0) = 5.1$ mol/L.
- In the pit solution $\big(0 < z < z_d(0)\big)$: $C(z,0) =$ linear profile from 10^{-6} mol/L at $z = 0$ to 5.1 mol/L at $z = z_d(0)$.

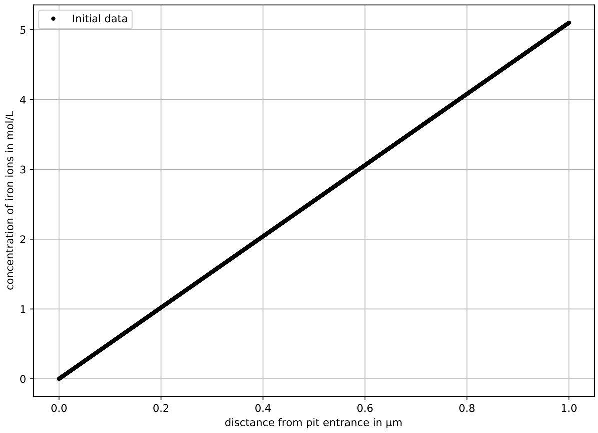


Figure 2. Initial distribution (at t=0) of the concentration of iron ions in the pit solution for initial pit depth $z_d(0) = 1$ μm.

Figure 3 illustrates the evolution of the pit depth for several values of the diffusion coefficient. We show that the depth increases when D increases. Indeed, we show that after Tf $= 1000$ hours of pit propagation, if the diffusion coefficient increases by factor λ, the pit depth increases $\approx \sqrt{\lambda}$ times.

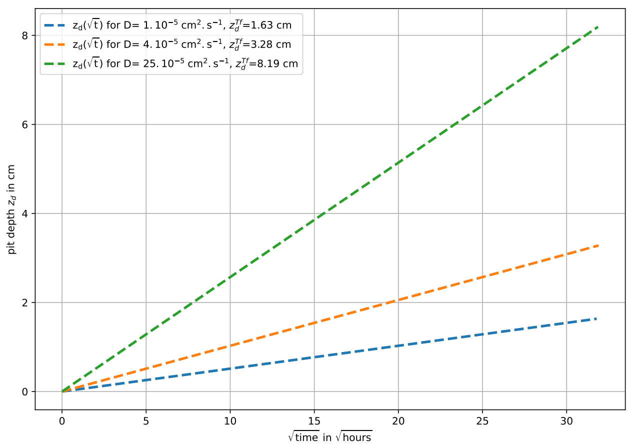


Figure 3. Evolution of the pit depth after 1000 hours of pit propagation for several values of the diffusion.

The values z_d^{Tf} are the final depths computed by the numerical code.

Remark 1.1 Let $\left(C_1, z_{d,1}\right)$ be a solution of Problem (1.1) for $D = 1$ and an initial concentration C_0, then

- $C_D(z,t) = C_1(z, Dt)$, $z_{d,D}(t) = z_{d,1}(Dt)$ is a solution of Problem (1.1).

- $\tilde{C}_D(z,t) = C_1\left(\frac{z}{\sqrt{D}}, t\right)$, $\tilde{z}_{d,D} = \sqrt{D}\ z_{d,1}(t)$ is a solution of Problem (1.1) with the initial condition $C_0\left(\frac{z}{\sqrt{D}}\right)$.

Effect of C_{sat} on the evolution of the pit depth :
In the case where $D = 0.85 \cdot 10^{-5}$ cm^2.s^{-1}, Figure 4 shows a comparison of the evolution of the pit depth during 1000 hours as a function of $\sqrt{\text{time}}$ (in $\sqrt{\text{hours}}$) for several values of C_{sat}.

The parameter C_{sat} has an influence on the evolution of the pit depth. Indeed, for a fixed time, the most important pit depth is the one computed for the largest value of C_{sat}.

Let us compare the two extremes values of C_{sat} where $C_{\text{sat},1} = 5.1$ mol/L and $C_{\text{sat},2} = 140.2$ mol/L after 1000 hours of pit propagation. For $C_{\text{sat},1} = 5.1$ mol/L, the pit depth is ≈ 1.51 cm while for $C_{\text{sat},2} = 140.2$ mol/L is ≈ 18.71 cm.

The ratio between the two depths is $\dfrac{18.71}{1.51} \approx 12.3$. Thus we deduce that even if the value of C_{sat} is very close to the value of C_{sol} (which is not realistic), the pit depth increases only 12 times comparing to the more realistic case where $C_{\text{sat}} = 5.1$ mol/L.

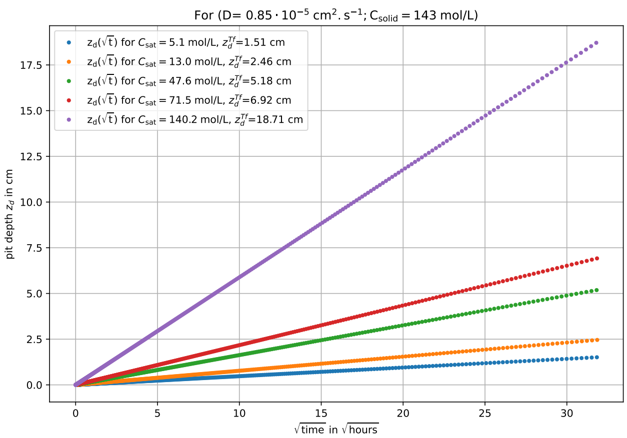


Figure 4. Evolution of the pit depth for several values of C_{sat} during 1000 hours.

Numerical simulations for the convergence to the self similar solution :
In this paragraph, we present some numerical results which illustrate the convergence of the solution (C, z_d) of Problem (1.1) to a special solution $(\widetilde{C}, \xi_d)$. Indeed, by means of a change of variables using the self-similar variable

$$\xi = \frac{z}{\sqrt{D(t+1)}}, \tag{1.3}$$

we can show (see [17] and also Section 5) that Problem (1.1) admits a self-similar solution $(\widetilde{C}, \xi_d)$ (a special solution) given by

$$\begin{cases} \widetilde{C}(\xi) = C_{\text{sat}} \dfrac{\int_0^{\xi} e^{-\frac{s^2}{4}} ds}{\int_0^{\xi_d} e^{-\frac{s^2}{4}} ds}, \\ 0 < \xi < \xi_d, \end{cases} \tag{1.4}$$

where ξ_d is the unique solution of the nonlinear equation

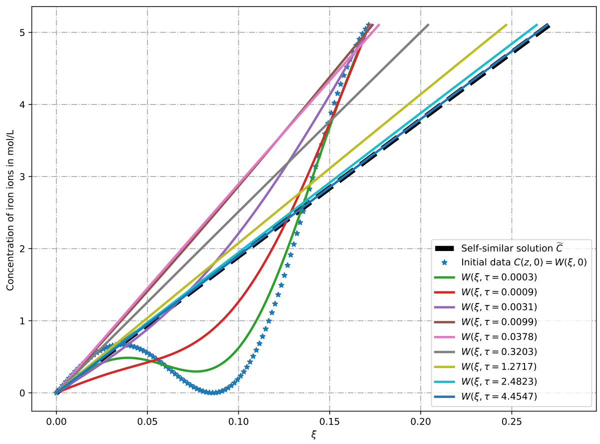
$$\frac{C_{\text{sat}}}{C_{\text{sol}} - C_{\text{sat}}} = \frac{\xi_d}{2} e^{\frac{\xi_d^2}{4}} \int_0^{\xi_d} e^{-\frac{s^2}{4}} ds. \tag{1.5}$$

Numerical simulations illustrate the convergence for long time of the solution (C, z_d) of Problem (1.1) to the self-similar solution $(\widetilde{C}, \xi_d)$. This convergence turns out to hold when starting from rather general initial conditions. We set

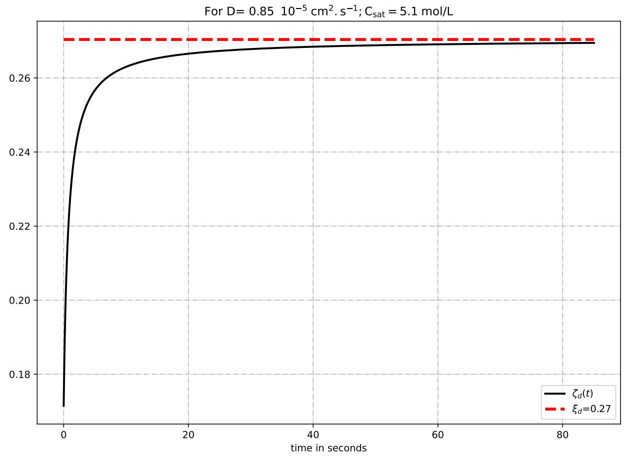
$$\begin{cases} W(\xi, \tau) = C(z, t) \ \text{ with } \ \tau = ln(t+1), \\ \zeta_d(t) = \dfrac{z_d(t)}{\sqrt{D(t+1)}}. \end{cases} \tag{1.6}$$

As an example, for $D = 0.85 \cdot 10^{-5}\,\text{cm}^2.\text{s}^{-1}$, $C_{\text{sat}} = 5.1\,\text{mol/L}$ and $C_{\text{sol}} = 143\,\text{mol/L}$, Figure 5 shows the convergence to the self-similar solution when starting from the initial data

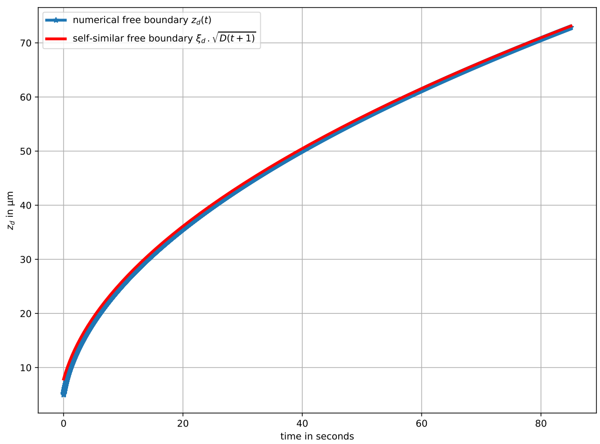
$$\begin{cases} z_d(0) = z_d^0 = 5\ \mu\text{m}, \\ C^0(z) := W(\xi, 0) = C(z, 0) = \dfrac{z\, C_{\text{sat}}}{2\, z_d^0} \left(1 + \sin\left(\dfrac{2\pi z}{z_d^0} + \dfrac{\pi}{2}\right)\right). \end{cases} \tag{1.7}$$



(a) Time evolution of the unknown function $W(\xi, \tau)$.



(b) Time evolution of the moving boundary $\zeta_d(t)$.



(c) Time evolution of the moving boundary $z_d(t)$.

Figure 5. Large time behavior of the solution.

The purpose of this article is to prove that under rather general hypotheses on the initial data, the solution of Problem (1.1) converges to the self-similar profile (1.4) $(\widetilde{C}, \xi_d)$ as $t \to +\infty$. We will do so for a slightly different formulation of the corrosion problem (1.1). In fact, by means of a change of variables, Problem (1.1) can be reformulated as the classical Stefan problem given in Avner Friedman's book on parabolic equations (Chapter 8 of [9, p.215]). To do so, we perform the change of variables

$$x = \frac{C_{\text{sol}} - C_{\text{sat}}}{\sqrt{D}}\, z, \quad s(t) = \frac{C_{\text{sol}} - C_{\text{sat}}}{\sqrt{D}}\, z_d(t) \;\text{ for all }\; t > 0, \tag{1.8}$$

and define

$$u(x,t) = C_{\text{sat}} - C(z,t) \;\text{ for all }\; t > 0, 0 \leqslant z \leqslant z_d(t) \;\text{ and }\; x = \frac{C_{\text{sol}} - C_{\text{sat}}}{\sqrt{D}}\, z. \tag{1.9}$$

Using the change of variables (1.8), Problem (1.1) becomes

$$\begin{cases} u_t = \alpha^2\, u_{xx}, & t > 0,\ 0 < x < s(t), \\ u(0,t) = C_{sat}, & t > 0, \\ u(s(t),t) = 0, & t > 0, \\ \dfrac{ds(t)}{dt} = -\alpha\, u_x(s(t),t), & t > 0, \end{cases} \tag{1.10}$$

where

$$\alpha = C_{\text{sol}} - C_{\text{sat}} > 0. \tag{1.11}$$

Setting $\alpha = 1$ and $C_{\text{sat}} = h$ in Problem (1.10) yields the problem which we study in this article.

2 Main results

We consider the problem

$$\begin{cases} u_t = u_{xx}, & t > 0, 0 < x < s(t), \\ u(0,t) = h, & t > 0, \\ u(s(t),t) = 0, & t > 0, \\ \dfrac{ds(t)}{dt} = -u_x(s(t),t), & t > 0, \\ s(0) = b_0, & \\ u(x,0) = u_0(x), & 0 < x < b_0 \end{cases} \tag{2.1}$$

where $x = s(t)$ is the unknown free boundary which is to be found together with $u(x,t)$.

In [9], Friedman proves that this problem has a unique smooth classical solution $\big(u(x,t), s(t)\big)$ in $Q := \{(x,t), t > 0, 0 < x < s(t)\}$. Moreover it follows from Schaeffer [15] and Friedman [7] that $s \in C^\infty(0,\infty)$ and that u is infinitely differentiable up to the free boundary s. We also refer to Fasano and Primicerio [6] for their study of related moving boundary problems.

The purpose of this paper is to study the large time behavior of the solution pair (u,s). Also let us mention some previous results from literature. Meirmanov [13] has proved that $\dfrac{s(t)}{\sqrt{t}} \to a$, where a is the unique solution of the nonlinear equation (2.4) below. Also, Ricci and Xie [14] have performed a stability analysis of some special solutions of a related one-phase Stefan problem on the semi-infinite interval $(s(t),\infty)$. In particular, they mention that the interface $s(t)$ behaves as $\beta\sqrt{t}$ for some positive constant β which they characterize. Moreover, Aiki and Muntean [1, 2], as mentioned by Zurek [21], have proved the existence of two positive constants c and C independent of t such that

$$c\,\sqrt{t} \leqslant s(t) \leqslant C\,\sqrt{t+1} \ \text{ for all } \ t \geqslant 0,$$

in the case of a more complicated system.

In this article, we will prove that the solution pair (u, s) converges to a self-similar solution as $t \to \infty$. First, let us define the self-similar solution. To do so, we introduce the self-similar variable

$$\eta = \frac{x}{\sqrt{t+1}}. \tag{2.2}$$

Then, the self-similar solution is given by

$$u(x,t) = U\left(\frac{x}{\sqrt{t+1}}\right) = U(\eta) = h\left[1 - \frac{\int_0^\eta e^{-\frac{s^2}{4}}\,ds}{\int_0^a e^{-\frac{s^2}{4}}\,ds}\right] \quad \text{for all } \eta \in (0,a), \tag{2.3}$$

where a is characterized as the unique solution of the nonlinear equation

$$h = \frac{a}{2} e^{\frac{a^2}{4}} \int_0^a e^{-\frac{s^2}{4}}\,ds. \tag{2.4}$$

In the first step, we will write the problem (2.1) in terms of η and t. To do so, we set

$$\begin{cases} V(\eta,t) = u(x,t), \\ a(t) = \dfrac{s(t)}{\sqrt{t+1}}. \end{cases} \tag{2.5}$$

However, the partial differential equation for V which we obtain explicitly involves the time variable t. It is given by

$$(t+1)V_t = V_{\eta\eta} + \frac{\eta}{2} V_\eta, \quad t > 0, \quad 0 < \eta < a(t). \tag{2.6}$$

This leads us to perform the change of time variable (see Hilhorst and Hulshof [10])

$$\tau = \ln(t+1), \tag{2.7}$$

and we set

$$\begin{cases} W(\eta,\tau) = V(\eta,t) = u(x,t), \\ b(\tau) = a(t) = \dfrac{s(t)}{\sqrt{t+1}}. \end{cases} \tag{2.8}$$

The full time evolution problem corresponding to the system (2.1) in coordinates η and τ is given by

$$\begin{cases} W_\tau = W_{\eta\eta} + \dfrac{\eta}{2} W_\eta, & \tau > 0, \quad 0 < \eta < b(\tau), \\ W(0,\tau) = h, & \tau > 0, \\ W(b(\tau),\tau) = 0, & \tau > 0, \\ \dfrac{db(\tau)}{d\tau} + \dfrac{b(\tau)}{2} = -W_\eta(b(\tau),\tau), & \tau > 0, \\ b(0) = b_0, & \\ W(\eta,0) = u_0(\eta), & 0 < \eta < b_0. \end{cases} \tag{2.9}$$

It is in the coordinates η and τ that we will rigorously characterize the large time behavior of the solution pair (W, b). However, for technical reasons, we sometimes have to use different variables, namely (y, τ) with $y = \dfrac{\eta}{b(\tau)}$ for all $0 < \eta < b(\tau)$. The use of the y variable allows to transform the problem (2.9) into a problem posed on a fixed domain.

To state an exact formulation of the results of this article, it is most convenient to use the variable y lying in $[0,1]$. In the variables (y,τ), the problem for $\big(\hat{W}(y,\tau), b(\tau)\big) = \big(W(\eta,\tau), b(\tau)\big)$ is given by

$$\begin{cases} \hat{W}_\tau(y,\tau) = \dfrac{1}{b^2(\tau)}\hat{W}_{yy}(y,\tau) + y\left(\dfrac{d\ln\big(b(\tau)\big)}{d\tau} + \dfrac{1}{2}\right)\hat{W}_y(y,\tau), & \tau > 0, \;\; 0 < y < 1, \\ \hat{W}(0,\tau) = h, & \tau > 0, \\ \hat{W}(1,\tau) = 0, & \tau > 0, \\ \dfrac{1}{2}\dfrac{db^2(\tau)}{d\tau} + \dfrac{b^2(\tau)}{2} = -\hat{W}_y(1,\tau), & \tau > 0, \\ b(0) = b_0, & \\ \hat{W}(y,0) = u_0(b_0 y), & 0 \leqslant y \leqslant 1. \end{cases} \tag{2.10}$$

The main result of this article is the following. We suppose that the initial data (u_0, b_0) satisfies the hypothesis:

$\mathbf{H}_0$: $u_0 \in C[0,\infty) \cap \mathbb{W}^{1,\infty}(0,b_0)$ with $u_0(0) = h$, $u_0(x) \geqslant 0$ for $0 \leqslant x \leqslant b_0$ and $u_0(x) = 0$ for all $x \geqslant b_0$.

Main Theorem 2.1 *Suppose that (u_0, b_0) satisfies the hypothesis $\boldsymbol{H}_0$. The unique solution $(\hat{W}, b)$ of Problem* (2.10) *is such that*

$$\lim_{\tau\to+\infty} ||\hat{W}(.,\tau) - \hat{U}||_{C^{1+\alpha}([0,1])} = 0 \quad \textit{for all} \;\; \alpha \in (0,1), \tag{2.11}$$

$$\lim_{\tau\to+\infty} b(\tau) = a, \tag{2.12}$$

where $(\hat{U}, a)$ is the unique solution of the stationary problem

$$\begin{cases} \dfrac{1}{a^2}\hat{U}_{yy} + \dfrac{y}{2}\hat{U}_y = 0, & 0 < y < 1, \\ \hat{U}(0) = h, \quad \hat{U}(1) = 0, & \\ \dfrac{a^2}{2} = -\hat{U}_y(1), & \end{cases} \tag{2.13}$$

which is equivalent to the stationary problem corresponding to Problem (2.9)

$$\begin{cases} U_{\eta\eta} + \dfrac{\eta}{2}U_\eta = 0, & 0 < \eta < a, \\ U(0) = h, \quad U(a) = 0, & \\ \dfrac{a}{2} = -U_\eta(a), & \end{cases} \tag{2.14}$$

for the self-similar solution of Problem (2.1).

Remark 2.2 The property (2.12) is equivalent to the convergence result

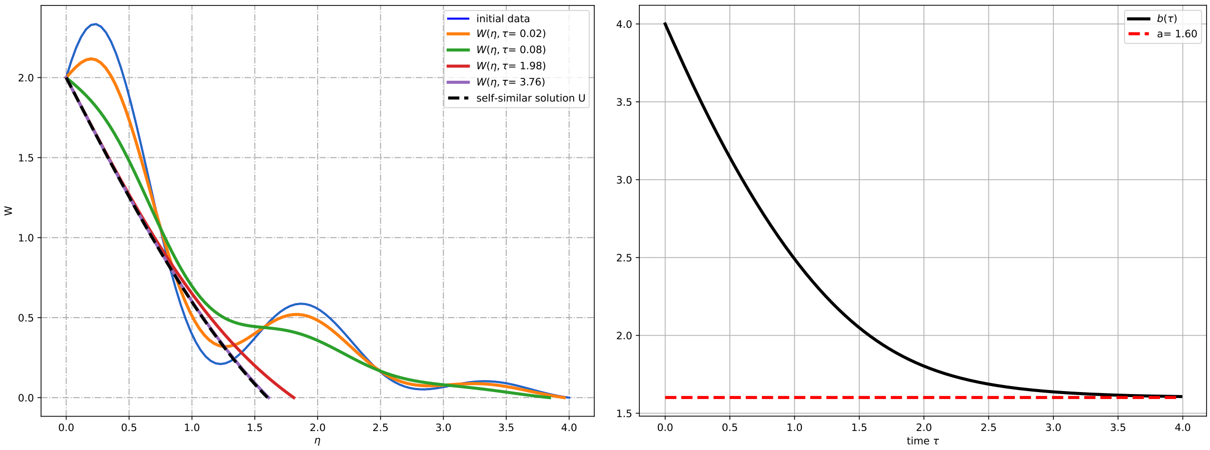
$$\frac{s(t)}{\sqrt{t+1}} \to a \;\text{ as }\; t \to +\infty, \tag{2.15}$$

which was already proved by Meirmanov [13].

We present in Figure 6 a numerical computation showing the large time behavior of the solution pair (W, b) defined in (2.9). The initial data (u_0, b_0) is chosen as follows

$$\begin{cases} b_0 = 4, \\ u_0(x) = h\left(\left(\dfrac{1}{1+x} - \dfrac{1}{1+b_0}\right)\dfrac{1+b_0}{b_0}\right)\dfrac{\sin\left(\dfrac{5x\pi}{b_0}\right)+1.5}{1.5} \end{cases} \tag{2.16}$$

with $h = 2$.



(a) Time evolution of the unknown function $W(\eta, \tau)$. (b) Time evolution of the moving boundary $b(\tau)$.

Figure 6. Large time behavior of the solution pair (W, b).

Organization of the paper :

In section 3, we introduce the Stefan problem given by Friedman [8] and recall known well-posedness and regularity results [7, 15]. Using a maximum principle [9], we show that if u_0 is nonnegative and bounded then the solution u is also nonnegative and bounded.

In Section 4, we start by defining a notion of upper and lower solutions for Problem (2.1). Then, we present a comparison principle in the (x, t) coordinates for a pair of upper and lower solutions of Problem (2.1).

In Section 5, we construct the self-similar solution (U, a). We will show that U is as given by (2.3) and a is characterized as the unique solution of the nonlinear equation (2.4).

In Section 6, we transform Problem (2.1) in coordinates (x, t) to obtain an equivalent problem, Problem (2.9), in coordinates (η, τ) where the solution pair becomes (W, b). We present an equivalent comparison principle in these coordinates and a class of functions which include both the lower and upper-solutions. We use the notation $(\bar{\mathcal{W}}, \bar{b})$ for the upper-solution, respectively $(\underline{\mathcal{W}}_\lambda, \underline{b}_\lambda)$ for the lower-solution depending on a parameter

λ. We also denote by $(W(\eta, \tau, (u_0, b_0)), b(\tau, (u_0, b_0)))$ the solution pair of Problem (2.9) with the initial conditions (u_0, b_0).
In Section 6, we also discuss some properties of upper and lower solutions. Then, we prove the monotonicity in time of the solution pair (W, b) of the time evolution Problem (2.9) with the two initial conditions $(\bar{\mathcal{W}}, \bar{b})$ and $(\underline{\mathcal{W}}_\lambda, \underline{b}_\lambda)$. In other words, we show that starting from a lower solution, the solution $\underline{W}(\eta, \tau) := W\big(\eta, \tau, (\underline{\mathcal{W}}_\lambda, \underline{b}_\lambda)\big)$ (i.e. with the initial conditions $(\underline{\mathcal{W}}_\lambda, \underline{b}_\lambda)$) increases in time as $\tau \to \infty$ to a limit function ψ and the corresponding moving boundary $\underline{b}(\tau) := b\big(\tau, (\underline{\mathcal{W}}_\lambda, \underline{b}_\lambda)\big)$ increases to a limit $\underline{b}_\infty$. Similarly, one can show that starting from an upper solution, the solution decreases to another limit function ϕ as $\tau \to \infty$ and the moving boundary $\bar{b}$ converges to a limit $\bar{b}_\infty$. However, we do not know yet whether ψ and ϕ coincide with the self-similar profile U and whether $\underline{b}_\infty$ and $\bar{b}_\infty$ coincide with the point a. In order to prove these results we first have to show extra a priori estimates which we do in the following section.

In Section 7, we prove a number of a priori estimates in the fixed domain. Indeed, we pass to fixed domain $(y, \tau) \in (0, 1) \times \mathbb{R}^+$ to avoid technical problems related to the characterization of the limits $\underline{b}_\infty$ and $\bar{b}_\infty$. In other words, we need to show that $\underline{W}_\eta(\underline{b}(\tau), \tau)$ converges to $\psi_\eta(\underline{b}_\infty)$ as $\tau \to \infty$. This requests to prove the uniform convergence of $\underline{W}_\eta(\eta, \tau)$ to its limit as $\tau \to \infty$ which we can more easily do in the fix domain coordinates.

Section 8 is devoted to the study of the limits as $\tau \to \infty$. More precisely, we prove that $(\psi, \underline{b}_\infty)$ verifies the following conditions

$$\psi(0) = h, \quad \psi(\underline{b}_\infty) = 0, \quad \frac{\underline{b}_\infty}{2} = -\psi_\eta(\underline{b}_\infty). \tag{2.17}$$

and ψ satisfies the ordinary differential equation

$$\psi_{\eta\eta} + \frac{\eta}{2}\psi_\eta = 0. \tag{2.18}$$

Similarly, it turns out that $\Big(W\big(\eta, \tau, (\bar{\mathcal{W}}, \bar{b})\big), b\big(\tau, (\bar{\mathcal{W}}, \bar{b})\big)\Big)$ converges as $\tau \to \infty$ towards the unique stationary solution $(\phi, \bar{b}_\infty)$ of Problem (2.9). At the end of Section 8, we show that the solution pair $(\psi, \underline{b}_\infty)$ coincides with the unique solution (U, a) of Problem (5.4) which also coincides with the solution pair $(\phi, \bar{b}_\infty)$.

3 Friedman's formulation

Let $h > 0$, $b > 0$. We define the function space

$$X^h(b) := \{u_0(x) \in C[0, \infty), \quad u_0(0) = h, \ u_0(x) \geqslant 0 \text{ for } 0 \leqslant x < b, \\ u_0(x) = 0 \text{ for } x \geqslant b\} \tag{3.1}$$

and we consider the problem

$$\begin{cases} u_t = u_{xx}, & t > 0, 0 < x < s(t), \\ u(0,t) = h, & t > 0, \\ u(s(t),t) = 0, & t > 0, \\ \dfrac{ds(t)}{dt} = -u_x(s(t),t), & t > 0, \\ s(0) = b_0, \\ u(x,0) = u_0(x) \in X^h(b_0). \end{cases} \tag{3.2}$$

Problem (3.2) is a free boundary problem where $x = s(t)$ is the free boundary to be found together with the unknown function $u(x,t)$.

Definition 3.1 *Let $T > 0$. We say that the pair (u,s) is a classical solution of Problem (3.2) if*

(1) *$s(t)$ is continuously differentiable for $0 \leqslant t \leqslant T$;*
(2) *$u \in C(\overline{Q_T})$, where $Q_T := \{(x,t), t \in (0,T], 0 < x < s(t)\}$;*
(3) *$u \in C^{2,1}(Q_T)$;*
(4) *$u_x \in C(\overline{Q_T^\delta})$ for all $\delta > 0$ where $Q_T^\delta = \{(x,t), t \in (\delta,T], 0 < x < s(t)\}$;*
(5) *the equations of Problem (3.2) are satisfied.*

Let $(u(x,t), s(t))$ be a solution of (3.2) for all $0 \leqslant t \leqslant T$. We extend u by:

$$u(x,t) = 0 \text{ for } x \geqslant s(t), \tag{3.3}$$

so that $u(\cdot,t)$ is defined for all $x \geqslant 0$.

Theorem 3.2 ([8, Theorem 1]) *Let $h > 0, b > 0$ and $u_0 \in X^h(b)$. Then, there exists a unique solution $(u(x,t), s(t))$ of (3.2) for all $t > 0$ in the classical sense. Moreover, the solution (u,s) is such that s is infinitely differentiable on $(0,\infty)$ and u is infinitely differentiable up to the free boundary for all $t > 0$ (see [7],[15]). Furthermore, the function $s(t)$ is strictly increasing in t.*

Proposition 3.3 *Let $h > 0, b > 0$, $\bar{h} > h$ and $u_0 \in X^h(b)$ such that $0 \leqslant u_0 \leqslant \bar{h}$. Then, the solution $(u(x,t), s(t))$ of (3.2) is such that $0 \leqslant u(x,t) \leqslant \bar{h}$ for all $(x,t) \in Q_T$.*

Proof. We apply the strong maximum principle (Theorem 1 of [9, p.34]) which states that if u attains its minimum or its maximum in an interior point $(x^0,t^0) \in Q_T$, then u is constant in Q_{t^0}. However, since $u(0,t) = h > 0$ for $t \in (0,T]$ and $u\big(s(t),t\big) = 0$, $u(.,t)$ cannot be constant in space on $\big(0,s(t)\big)$, so that u attains its minimum and its maximum on the boundary $\Gamma := \{(0,t), 0 \leqslant t \leqslant T\} \cup \{(x,0), 0 < x < b\} \cup \{(s(t),t), 0 \leqslant t \leqslant T\}$. As $0 \leqslant u_0 \leqslant \bar{h}$, we conclude that $0 \leqslant u(x,t) \leqslant \bar{h}$ for all $(x,t) \in Q_T$. □

4 Comparison principle

To begin with, we define the lower and upper solutions of Problem (3.2), which permits to bound the solution pair (u, s) from below and from above.

Definition 4.1 *For $u \in C(\overline{Q_T}) \cap C^{2,1}(Q_T)$, we define $\mathcal{L}(u) := u_t - u_{xx}$. The pair $(\underline{u}, \underline{s})$ is a lower solution of the Problem (3.2) if it satisfies*

$$\begin{cases} \mathcal{L}(\underline{u}) = \underline{u}_t - \underline{u}_{xx} \leqslant 0 \ \text{ in } \mathrm{Q_T}, \\ \underline{u}(0,t) \leqslant h, \quad \underline{u}(\underline{s}(t),t) = 0, & t > 0, \\ \dfrac{d\underline{s}(t)}{dt} \leqslant -\underline{u}_x(\underline{s}(t),t), & t > 0, \\ \underline{s}(0) \leqslant b_0, \\ \underline{u}(x,0) \leqslant u_0(x), & x \in (0, b_0). \end{cases} \tag{4.1}$$

The pair $(\bar{u}, \bar{s})$ is an upper solution of the Problem (3.2) if it satisfies (4.1) *with all $\leqslant$ replaced by $\geqslant$.*

Theorem 4.2 (Comparison principle) *Let $(u_1(x,t), s_1(t))$ and $(u_2(x,t), s_2(t))$ be respectively lower and upper solutions of (3.2) corresponding respectively to the data (h_1, u_{01}, b_1) and (h_2, u_{02}, b_2).*
If $b_1 \leqslant b_2$, $h_1 \leqslant h_2$ and $u_{01} \leqslant u_{02}$, then $s_1(t) \leqslant s_2(t)$ for $t \geqslant 0$ and $u_1(x,t) \leqslant u_2(x,t)$ for $x \geqslant 0$ and $t \geqslant 0$.

Proof of Theorem 4.2. The proof is rather similar to those presented by Du & Lou [4, Lemma 2.2 and Remark 2.3] and Du & Lin [5, Lemma 3.5]. We omit it here. □

5 Self-similar solution

We now look for a self-similar solution of the problem

$$\begin{cases} u_t = u_{xx}, & t > 0, \quad 0 < x < s(t), \\ u(0,t) = h, & t > 0, \\ u(s(t),t) = 0, & t > 0, \\ \dfrac{ds(t)}{dt} = -u_x(s(t),t), & t > 0, \end{cases} \tag{5.1}$$

in the form

$$\begin{cases} u(x,t) = U\left(\dfrac{x}{\sqrt{t+1}}\right), \\ s(t) = a\sqrt{t+1}, \end{cases} \tag{5.2}$$

for some positive constant a still to be determined. We set

$$\eta := \frac{x}{\sqrt{t+1}} \tag{5.3}$$

and deduce that

$$\begin{cases} U_{\eta\eta} + \dfrac{\eta}{2} U_\eta = 0, & 0 < \eta < a, \\ U(0) = h, & U(a) = 0. \end{cases} \tag{5.4}$$

The unique solution of (5.4) is given by

$$U(\eta) = h\left[1 - \frac{\int_0^\eta e^{-\frac{s^2}{4}}\,ds}{\int_0^a e^{-\frac{s^2}{4}}\,ds}\right] \quad \text{for all } \eta \in (0, a). \tag{5.5}$$

It remains to determine the constant a. We write that

$$s'(t) = \frac{a}{2\sqrt{t+1}} = -u_x(s(t), t) = -\frac{U_\eta\left(\dfrac{s(t)}{\sqrt{t+1}}\right)}{\sqrt{t+1}}, \tag{5.6}$$

which implies that

$$\frac{a}{2} = -U_\eta(a), \tag{5.7}$$

so that a is characterized as the unique solution of the equation

$$h = \frac{a}{2} e^{\frac{a^2}{4}} \int_0^a e^{-\frac{s^2}{4}}\,ds. \tag{5.8}$$

We remark that the function $a = a(h)$ is strictly increasing, which in turn implies that the functional $h \to U$ is strictly increasing.

We conclude that the self-similar solution of Problem (5.1) coincides with the unique solution (U, a) of Problem (2.14).
Finally, we remark that the self-similar solution given by (5.5) and (5.8) is a translation in time of the Lamé-Clapeyron solution [11] (see more details in Tarzia [20]).

6 New coordinates and construction of upper and lower solutions

We set

$$\begin{cases} V(\eta, t) = u(x, t), \\ a(t) = \dfrac{s(t)}{\sqrt{t+1}}, \end{cases} \tag{6.1}$$

with η given by (5.3). We obtain the problem

$$\begin{cases} (t+1)V_t = V_{\eta\eta} + \dfrac{\eta}{2} V_\eta, & t > 0, \quad 0 < \eta < a(t), \\ V(0, t) = h, \quad V(a(t), t) = 0, & t > 0, \\ (t+1)\dfrac{da(t)}{dt} + \dfrac{a(t)}{2} = -V_\eta(a(t), t), & t > 0. \end{cases} \tag{6.2}$$

Finally we set

$$\tau = \ln(t+1). \tag{6.3}$$

The equations in the system (6.2) read as

$$\begin{cases} W_\tau = W_{\eta\eta} + \dfrac{\eta}{2} W_\eta, & \tau > 0, \quad 0 < \eta < b(\tau), \\ W(0,\tau) = h, \quad W(b(\tau),\tau) = 0, & \tau > 0, \\ \dfrac{db(\tau)}{d\tau} + \dfrac{b(\tau)}{2} = -W_\eta(b(\tau),\tau), & \tau > 0, \end{cases} \tag{6.4}$$

where we have set

$$W(\eta,\tau) = V(\eta,t), \; b(\tau) = a(t). \tag{6.5}$$

Next, we write the full time evolution problem corresponding to the system (6.4). It is given by

$$\begin{cases} W_\tau = W_{\eta\eta} + \dfrac{\eta}{2} W_\eta, & \tau > 0, \quad 0 < \eta < b(\tau), \\ W(0,\tau) = h, & \tau > 0, \\ W(b(\tau),\tau) = 0, & \tau > 0, \\ \dfrac{db(\tau)}{d\tau} + \dfrac{b(\tau)}{2} = -W_\eta(b(\tau),\tau), & \tau > 0, \\ b(0) = b_0, & \\ W(\eta,0) = u_0(\eta), & 0 \leqslant \eta \leqslant b_0. \end{cases} \tag{6.6}$$

Finally, we note that the stationary solution of Problem (6.6) coincides with the unique solution of Problem (2.14), or in other words, the self-similar solution of Problem (2.1).

Definition 6.1 *We define the linear operator $\mathcal{L}(W) := W_\tau - W_{\eta\eta} - \dfrac{\eta}{2} W_\eta$. The pair $(\underline{W}, \underline{b})$ is a lower solution of Problem (6.6) if it satisfies:*

$$\begin{cases} \mathcal{L}(\underline{W}) = \underline{W}_\tau - \underline{W}_{\eta\eta} - \dfrac{\eta}{2} \underline{W}_\eta \leqslant 0, & \tau > 0, \quad 0 < \eta < \underline{b}(\tau), \\ \underline{W}(0,\tau) \leqslant h, \quad \underline{W}(\underline{b}(\tau),\tau) = 0, & \tau > 0, \\ \dfrac{d\underline{b}(\tau)}{d\tau} + \dfrac{\underline{b}(\tau)}{2} \leqslant -\underline{W}_\eta(\underline{b}(\tau),\tau), & \tau > 0, \\ \underline{b}(0) \leqslant b_0, & \\ \underline{W}(\eta,0) \leqslant u_0(\eta), & 0 \leqslant \eta \leqslant \underline{b}(0). \end{cases} \tag{6.7}$$

Similarly, $(\bar{W}, \bar{b})$ is an upper solution of the Problem (6.6) if it satisfies Problem (6.7) *with all $\leqslant$ replaced with $\geqslant$.*

Finally, one can deduce from Theorem 4.2 the following comparison principle.

Theorem 6.2 *Let $\big(W_1(\eta,\tau), b_1(\tau)\big)$ and $\big(W_2(\eta,\tau), b_2(\tau)\big)$ be respectively lower and upper solutions of (6.6) corresponding respectively to the data (h_1, u_{01}, b_{01}) and (h_2, u_{02}, b_{02}). If $b_{01} \leqslant b_{02}$, $h_1 \leqslant h_2$ and $u_{01} \leqslant u_{02}$, then $b_1(\tau) \leqslant b_2(\tau)$ for $\tau \geqslant 0$ and $W_1(\eta,\tau) \leqslant W_2(\eta,\tau)$ for $\eta \geqslant 0$ and $\tau \geqslant 0$.*

Throughout this paper, we will also make use of the explicit notation $W\big(\eta, \tau, (u_0, b_0)\big)$ and $b\big(\tau, (u_0, b_0)\big)$ for the solution pair associated with the initial data (u_0, b_0).

Construction of upper and lower solutions

Now, we construct ordered upper and lower solutions for Problem (6.6). Let (u_0, b_0) be the initial data satisfying the hypothesis $\mathbf{H}_0$ in Section 2.

Upper solution. Let $\bar{h} > h$. We consider (W_λ, b_λ) the unique solution of the problem

$$\begin{cases} W_{\eta\eta} + \dfrac{\lambda\eta}{2} W_\eta = 0, & 0 < \eta < b, \\ W(0) = \bar{h}, \quad W(b) = 0, & \\ \dfrac{b}{2} = -W_\eta(b), & \end{cases} \tag{6.8}$$

which is given by

$$W_\lambda(\eta) = \bar{h}\left[1 - \frac{\int_0^\eta e^{-\frac{\lambda s^2}{4}} ds}{\int_0^{b_\lambda} e^{-\frac{\lambda s^2}{4}} ds}\right] \quad \text{for all } \eta \in (0, b_\lambda) \tag{6.9}$$

and b_λ is the unique solution of the equation

$$\bar{h} = \frac{b_\lambda}{2} e^{\frac{\lambda b_\lambda^2}{4}} \int_0^{b_\lambda} e^{-\frac{\lambda s^2}{4}} ds. \tag{6.10}$$

We easily check that W_λ satisfies the following property

$$-W_{\lambda,\eta\eta} - \frac{\eta}{2} W_{\lambda,\eta} \geqslant 0 \quad \text{if and only if} \quad \lambda \leqslant 1. \tag{6.11}$$

Now, we suppose that

$$\lambda \leqslant 1, \tag{6.12}$$

and we define $(\bar{\mathcal{W}}_1, \bar{b})$ by

$$\bar{b} = b_\lambda \quad \text{and} \quad \bar{\mathcal{W}}_1(\eta) := W_\lambda(\eta) \;\text{ if } 0 \leqslant \eta \leqslant \bar{b}, \tag{6.13}$$

where $W_\lambda(\eta)$ is given by (6.9) and $\bar{b}$ satisfy the equation (6.10). Then $(\bar{\mathcal{W}}_1, \bar{b})$ with $\lambda \leqslant 1$, satisfies the following system

$$\begin{cases} -\bar{\mathcal{W}}_{1,\eta\eta} - \dfrac{\eta}{2} \bar{\mathcal{W}}_{1,\eta} \geqslant 0, & 0 < \eta < \bar{b}, \\ \bar{\mathcal{W}}_1(0) = \bar{h} \geqslant h, \quad \bar{\mathcal{W}}_1(\bar{b}) = 0, & \\ \dfrac{\bar{b}}{2} = -\bar{\mathcal{W}}_{1,\eta}(\bar{b}). & \end{cases} \tag{6.14}$$

Therefore, the pair $(\bar{\mathcal{W}}_1, \bar{b})$ with $\lambda \leqslant 1$, will be an upper solution of Problem (6.6) if $\bar{b} \geqslant b_0$ and $\bar{\mathcal{W}}_1 \geqslant u_0$ in $[0, \bar{b}]$.

Now, we consider the function $\bar{\mathcal{W}}_2$ solution of the following problem

$$\begin{cases} W_{\eta\eta} + \dfrac{\lambda\eta}{2} W_\eta = 0, & \eta > 0, \\ W(0) = h, \quad W_\eta(0) > 0, & \end{cases} \tag{6.15}$$

which is given by

$$\bar{\mathcal{W}}_2(\eta) = h + \bar{\mathcal{W}}_{2,\eta}(0) \int_0^\eta e^{-\frac{\lambda s^2}{4}} ds \ \text{ for all } \ \eta > 0, \tag{6.16}$$

where $\bar{\mathcal{W}}_{2,\eta}(0) > 0$.

We search λ such that $\bar{\mathcal{W}}_2$ satisfy the following inequality

$$-\bar{\mathcal{W}}_{2,\eta\eta} - \frac{\eta}{2}\bar{\mathcal{W}}_{2,\eta} \geqslant 0. \tag{6.17}$$

We have that

$$-\bar{\mathcal{W}}_{2,\eta\eta}(\eta) - \frac{\eta}{2}\bar{\mathcal{W}}_{2,\eta}(\eta) = \frac{\eta}{2}\bar{\mathcal{W}}_{2,\eta}(0)e^{-\frac{\lambda\eta^2}{4}}(\lambda - 1), \tag{6.18}$$

so that, if

$$\lambda \geqslant 1, \tag{6.19}$$

then (6.17) holds. So, we consider

$$\bar{\mathcal{W}}_2(\eta) = h + \bar{\mathcal{W}}_{2,\eta}(0) \int_0^\eta e^{-\frac{\lambda s^2}{4}} ds \ \text{ for all } \ \eta > 0 \ \text{ and } \ \lambda \geqslant 1. \tag{6.20}$$

For all $\tau \geqslant 0$, the function $\bar{\mathcal{W}}_2$ satisfies the problem

$$\begin{cases} -\bar{\mathcal{W}}_{2,\eta\eta} - \frac{\eta}{2}\bar{\mathcal{W}}_{2,\eta} \geqslant 0, \quad 0 < \eta < b(\tau), \\ \bar{\mathcal{W}}_2(0) = h, \quad \bar{\mathcal{W}}_2(b(\tau)) \geqslant 0. \end{cases} \tag{6.21}$$

We recall that we denote by $(W(\eta, \tau, (u_0, b_0)), b(\tau, (u_0, b_0)))$ the solution pair of Problem (6.6) with the initial conditions (u_0, b_0). According to the classical maximum principle for parabolic equations, we will deduce that

$$\bar{\mathcal{W}}_2(\eta) \geqslant W(\eta, \tau, (u_0, b_0)), \ \text{for all } \tau \geqslant 0, \eta \geqslant 0,$$

if we are able to prove that $\bar{\mathcal{W}}_2 \geqslant u_0$ in $[0, b_0]$.

Now we define the pair $(\bar{\mathcal{W}}, \bar{b})$ where

$$\bar{\mathcal{W}} := min(\bar{\mathcal{W}}_1, \bar{\mathcal{W}}_2) \ \text{ and } \ \bar{b} \ \text{ is given by (6.13).} \tag{6.22}$$

We will prove in the next Lemma that $\bar{\mathcal{W}}$ is bounded from below by u_0 in $[0, b_0]$.

We recall that the function space $X^h(b)$ is defined in (3.1).

Lemma 6.3 *Let $u_0 \in X^h(b_0) \cap \mathbb{W}^{1,\infty}\big(0, b_0\big)$. The pair $(\bar{\mathcal{W}}, \bar{b})$ defined in (6.22) is such that $u_0 \leqslant \bar{\mathcal{W}}$ in $[0, b_0]$ and $b_0 \leqslant \bar{b}$. Moreover, we have that*

$$b\big(\tau, (u_0, b_0)\big) \leqslant \bar{b} \ \text{ and } \ W\big(\eta, \tau, (u_0, b_0)\big) \leqslant \bar{\mathcal{W}}(\eta) \text{ for all } \tau \geqslant 0, \eta \geqslant 0, \tag{6.23}$$

where $(W(\eta, \tau, (u_0, b_0)), b(\tau, (u_0, b_0)))$ denotes the solution pair of Problem (6.6) with the initial conditions (u_0, b_0).

Proof. Define

$$M := \max\left(\|\frac{du_0}{d\eta}\|_{L^\infty(0,b_0)}, \frac{h}{b_0}, \frac{b_0}{2}\right). \tag{6.24}$$

From the equalities

$$u_0(\eta) = h + \int_0^\eta \frac{du_0}{d\eta}(s)\,ds \text{ and } u_0(\eta) = -\int_\eta^{b_0} \frac{du_0}{d\eta}(s)\,ds \text{ for } 0 \leqslant \eta \leqslant b_0,$$

we deduce that

$$u_0 \leqslant \min\big(h + M\eta, M(b_0 - \eta)\big) \text{ for all } \eta \in (0, b_0). \tag{6.25}$$

We define

$$\bar{h} = Mb_0, \tag{6.26}$$

and

$$\bar{\mathcal{W}}_1(\eta) = \bar{h}\big(1 - \frac{\eta}{\bar{b}}\big) = Mb_0\big(1 - \frac{\eta}{\bar{b}}\big) \text{ for all } 0 < \eta < \bar{b}, \tag{6.27}$$

with

$$\bar{b} = \sqrt{2\bar{h}}, \tag{6.28}$$

so that

$$\bar{b} = \sqrt{2Mb_0}. \tag{6.29}$$

The pair $(\bar{\mathcal{W}}_1, \bar{b})$ is an upper solution such that

$$\bar{\mathcal{W}}_1 \geqslant u_0 \text{ in } [0, b_0]. \tag{6.30}$$

Then, according to the comparaison principle Theorem 6.2, it follows that

$$b\big(\tau, (u_0, b_0)\big) \leqslant \bar{b} \text{ and } \quad W\big(\eta, \tau, (u_0, b_0)\big) \leqslant \bar{\mathcal{W}}_1(\eta) \text{ for all } \tau \geqslant 0, \eta \geqslant 0. \tag{6.31}$$

Now, we turn to $\bar{\mathcal{W}}_2$. In view of (6.20), we recall that

$$\bar{\mathcal{W}}_2(\eta) = h + \bar{\mathcal{W}}_{2,\eta}(0) \int_0^\eta e^{-\frac{\lambda s^2}{4}}\,ds \text{ for all } \eta > 0 \text{ and } \lambda \geqslant 1.$$

In particular

$$\bar{\mathcal{W}}_2(0) = h. \tag{6.32}$$

Next, we compute the coordinates of the intersection point between the lines

$$\zeta = h + M\eta \text{ for all } \eta > 0 \text{ and } \bar{\mathcal{W}}_1 : \zeta = Mb_0\left(1 - \frac{\eta}{\sqrt{2Mb_0}}\right) \text{ for all } 0 < \eta < \bar{b}. \tag{6.33}$$

We note the intersection point by $P = (\eta_p, \zeta_p)$.

We have that

$$h + M\eta_p = Mb_0\left(1 - \frac{\eta_p}{\sqrt{2Mb_0}}\right), \tag{6.34}$$

so that

$$\eta_p\left(M + \frac{Mb_0}{\sqrt{2Mb_0}}\right) = Mb_0 - h. \tag{6.35}$$

In view of (6.35), it follows that

$$\eta_p = \frac{Mb_0 - h}{M + \sqrt{\dfrac{Mb_0}{2}}}. \tag{6.36}$$

In view of (6.33), we deduce that

$$P = (\eta_p, \zeta_p) = \left(\frac{Mb_0 - h}{M + \sqrt{\dfrac{Mb_0}{2}}},\ h + \frac{Mb_0 - h}{1 + \sqrt{\dfrac{b_0}{2M}}} \right). \tag{6.37}$$

Next, we write that

$$\bar{\mathcal{W}}_2(\eta_p) = \zeta_p, \tag{6.38}$$

that is

$$h + \bar{\mathcal{W}}_{2,\eta}(0) \int_0^{\eta_p} e^{-\frac{\lambda s^2}{4}} ds = \zeta_p \ \text{ for all } \ \lambda \geqslant 1. \tag{6.39}$$

Thus, we deduce that

$$\bar{\mathcal{W}}_{2,\eta}(0) = \frac{\zeta_p - h}{\int_0^{\eta_p} e^{-\frac{\lambda s^2}{4}} ds} \ \text{ for all } \ \lambda \geqslant 1. \tag{6.40}$$

In view of (6.40), we obtain

$$\bar{\mathcal{W}}_2(\eta) = h + \frac{\zeta_p - h}{\int_0^{\eta_p} e^{-\frac{\lambda s^2}{4}} ds} \int_0^{\eta} e^{-\frac{\lambda s^2}{4}} ds \ \text{ for all } \ \eta > 0 \ \text{ and } \ \lambda \geqslant 1. \tag{6.41}$$

We can easily check that

$$\bar{\mathcal{W}}_2 \geqslant u_0 \text{ in } [0, b_0]. \tag{6.42}$$

Then, according to the classical maximum principle (see [Protter-Weinberger]), we deduce that

$$\bar{\mathcal{W}}_2(\eta) \geqslant W(\eta, \tau, (u_0, b_0)), \ \text{ for all } \tau \geqslant 0, \eta \geqslant 0. \tag{6.43}$$

Now, in view of (6.30), (6.31), (6.42) and (6.43), it follows that

$$\bar{\mathcal{W}} := min(\bar{\mathcal{W}}_1, \bar{\mathcal{W}}_2) \geqslant u_0 \text{ in } [0, b_0] \ \text{ and } \ \bar{b} \geqslant b_0 \tag{6.44}$$

and the comparison estimates (6.23) hold.
This completes the proof of Lemma 6.3. □

Lower solution. We consider (W_λ, b_λ) the unique solution of the problem

$$\begin{cases} W_{\eta\eta} + \dfrac{\lambda\eta}{2} W_\eta = 0, & 0 < \eta < b, \\ W(0) = h, \quad W(b) = 0, & \\ \dfrac{b}{2} = -W_\eta(b), & \end{cases} \tag{6.45}$$

which is given by

$$W_\lambda(\eta) = h\left[1 - \frac{\int_0^{\eta} e^{-\frac{\lambda s^2}{4}} ds}{\int_0^{b_\lambda} e^{-\frac{\lambda s^2}{4}} ds}\right] \quad \text{for all } \eta \in (0, b_\lambda) \tag{6.46}$$

and b_λ is the unique solution of the equation

$$h = \frac{b_\lambda}{2} e^{\frac{\lambda b_\lambda^2}{4}} \int_0^{b_\lambda} e^{-\frac{\lambda s^2}{4}} ds. \tag{6.47}$$

We can easily show the following properties for (W_λ, b_λ).

Lemma 6.4 *We have that*

$$0 \leqslant W_\lambda(\eta) \leqslant h \quad \textit{for all } \lambda \geqslant 0 \textit{ and } 0 \leqslant \eta \leqslant b_\lambda, \tag{6.48}$$

$$W_{\lambda,\eta}(\eta) \leqslant 0 \quad \textit{for all } \lambda \geqslant 0 \textit{ and } 0 \leqslant \eta \leqslant b_\lambda \tag{6.49}$$

and

$$W_{\lambda,\eta\eta}(\eta) \geqslant 0 \quad \textit{for all } \lambda \geqslant 0 \textit{ and } 0 \leqslant \eta \leqslant b_\lambda. \tag{6.50}$$

In particular,

$$W_\lambda \ \textit{ is } \begin{cases} \textit{a linear function} & \textit{if } \lambda = 0, \\ \textit{a convex function} & \textit{if } \lambda > 0. \end{cases} \tag{6.51}$$

We easily check that W_λ satisfies the following property

$$-W_{\lambda,\eta\eta} - \frac{\eta}{2} W_{\lambda,\eta} \leqslant 0 \quad \text{ if and only if } \quad \lambda \geqslant 1. \tag{6.52}$$

Now, we suppose that

$$\lambda \geqslant 1, \tag{6.53}$$

and we define $(\underline{W}_\lambda, \underline{b}_\lambda)$ by

$$\underline{b}_\lambda = b_\lambda \quad \text{and} \quad \underline{W}_\lambda(\eta) := \begin{cases} W_\lambda(\eta) & \text{if } 0 \leqslant \eta \leqslant \underline{b}_\lambda, \\ 0 & \text{if } \eta > \underline{b}_\lambda, \end{cases} \tag{6.54}$$

where $W_\lambda(\eta)$ is given by (6.46) and $\underline{b}_\lambda$ satisfies the equation (6.47). The pair $(\underline{W}_\lambda, \underline{b}_\lambda)$ will be a lower solution of Problem (6.6) if $\underline{b}_\lambda \leqslant b_0$ and $\underline{W}_\lambda \leqslant u_0$ in $[0, b_0]$.

Next we establish some further properties for the free boundary position b_λ.

Lemma 6.5 *The following properties hold for b_λ satisfying* (6.47).

(1) b_λ *is a decreasing function of* λ.

(2) $b_\lambda \to 0$ *as* $\lambda \to +\infty$.

Proof. We start to prove (i). We define $\mathcal{F}$ as the function given by

$$\mathcal{F}(\lambda, b_\lambda) = \frac{b_\lambda}{2} \int_0^{b_\lambda} e^{\frac{\lambda(b_\lambda^2 - s^2)}{4}} ds - h \tag{6.55}$$

and consider the equation $\mathcal{F}(\lambda, b_\lambda) = 0$. We compute the differential of $\mathcal{F}$ through partial derivatives given by

$$d\mathcal{F} = \frac{\partial \mathcal{F}}{\partial \lambda} d\lambda + \frac{\partial \mathcal{F}}{\partial b_\lambda} db_\lambda. \tag{6.56}$$

From (6.55), it follows that

$$\frac{\partial \mathcal{F}}{\partial \lambda} = \frac{b_\lambda}{2} \int_0^{b_\lambda} \frac{(b_\lambda^2 - s^2)}{4} e^{\frac{\lambda(b_\lambda^2 - s^2)}{4}} ds \quad > 0 \ \text{ for all } \ b_\lambda > 0 \tag{6.57}$$

and

$$\frac{\partial \mathcal{F}}{\partial b_\lambda} = \frac{1}{2}\int_0^{b_\lambda} e^{\frac{\lambda(b_\lambda^2 - s^2)}{4}} ds + \frac{b_\lambda}{2}\left(1+\int_0^{b_\lambda} \frac{2b_\lambda\lambda}{4} e^{\frac{\lambda(b_\lambda^2 - s^2)}{4}} ds\right) \quad > 0 \text{ for all } b_\lambda > 0. \tag{6.58}$$

Since $\mathcal{F}(\lambda, b_\lambda) = 0$, it follows from (6.56) that

$$\frac{\partial \mathcal{F}(\lambda, b_\lambda)}{\partial \lambda} d\lambda + \frac{\partial \mathcal{F}(\lambda, b_\lambda)}{\partial b_\lambda} db_\lambda = 0. \tag{6.59}$$

Thus, since $\dfrac{\partial \mathcal{F}}{\partial b_\lambda} \neq 0$, it follows from (6.57),(6.58) and (6.59) that

$$\frac{db_\lambda}{d\lambda} = -\frac{\dfrac{\partial \mathcal{F}(\lambda, b_\lambda)}{\partial \lambda}}{\dfrac{\partial \mathcal{F}(\lambda, b_\lambda)}{\partial b_\lambda}} \quad < 0, \tag{6.60}$$

which completes the proof of (i).

Now, we turn to the proof of (ii). For $\lambda \geqslant 0$, we have $b_\lambda > 0$ and b_λ is a decreasing function of λ. Hence, there exists $\alpha \geqslant 0$ such that $b_\lambda \to \alpha$ as $\lambda \to +\infty$ and $b_\lambda \geqslant \alpha$ for all $\lambda \geqslant 0$. We shall prove that $\alpha = 0$. This fact mainly relies on the following inequality which will be proved later on. Let $a \geqslant 0$. For $\lambda \geqslant 0$ large enough, the following inequality holds :

$$\int_0^a e^{-\frac{\lambda s^2}{4}} ds \geqslant a(1 + \frac{\lambda}{4}a^2)e^{-\frac{\lambda a^2}{4}}. \tag{6.61}$$

Since $b_\lambda \geqslant \alpha$ for all $\lambda \geqslant 0$, we deduce from (6.47) that

$$h \geqslant \frac{\alpha}{2} e^{\frac{\lambda\alpha^2}{4}} \int_0^\alpha e^{-\frac{\lambda s^2}{4}} ds. \tag{6.62}$$

For λ large enough we infer from the estimate (6.61) that

$$h \geqslant \frac{\alpha^2}{2}(1 + \frac{\lambda}{4}\alpha^2). \tag{6.63}$$

Letting $\lambda \to +\infty$ in (6.63), we see that we necessarily have $\alpha = 0$. It remains to prove that the inequality (6.61) holds for λ large enough. We only have to consider the case where $a > 0$ since (6.61) is trivially true for $a = 0$. Let us introduce $f(x) = e^{-\frac{\lambda x^2}{4}}$. We have $f''(x) = \frac{\lambda}{2}(\frac{\lambda}{2}x^2 - 1)e^{-\frac{\lambda x^2}{4}}$. We choose $\lambda > 0$ large enough to have $0 < \sqrt{\frac{2}{\lambda}} < a$ and then f is convex in $\big[\sqrt{\frac{2}{\lambda}}, a\big]$. Therefore, for all $x \in \big[\sqrt{\frac{2}{\lambda}}, a\big]$ we have

$$f(x) \geqslant g(x) := f(a) + (x - a)f'(a) \tag{6.64}$$

that is

$$e^{-\frac{\lambda x^2}{4}} \geqslant \left(1 + \frac{\lambda}{2}a(a - x)\right) e^{-\frac{\lambda a^2}{4}}, \quad \text{for all } x \in \Big[\sqrt{\frac{2}{\lambda}}, a\Big]. \tag{6.65}$$

Next we prove that (6.64) also holds for $x \in \big[0, \sqrt{\frac{2}{\lambda}}\big]$. Indeed, we have

$$\max_{x\in\left[0,\sqrt{\frac{2}{\lambda}}\right]} g(x) = g(0) = (1 + \frac{\lambda}{2}a^2)e^{-\frac{\lambda a^2}{4}}$$

and

$$\min_{x\in[0,\sqrt{\frac{2}{\lambda}}]} f(x) = f(\sqrt{\frac{2}{\lambda}}) = e^{-\frac{1}{2}}.$$

Since $g(0) \to 0$ as $\lambda \to +\infty$, we get, for λ large enough

$$\max_{[0,\sqrt{\frac{2}{\lambda}}]} g = g(0) \leqslant \min_{[0,\sqrt{\frac{2}{\lambda}}]} f = e^{-\frac{1}{2}} \tag{6.66}$$

and then

$$g(x) \leqslant f(x), \text{ for all } x \in \left[0, \sqrt{\frac{2}{\lambda}}\right] \tag{6.67}$$

Combining (6.64) with (6.67) leads to $f(x) \geqslant g(x)$ for all $x \in [0,a]$, that is

$$e^{-\frac{\lambda x^2}{4}} \geqslant \left(1 + \frac{\lambda}{2}a(a-x)\right) e^{-\frac{\lambda a^2}{4}}, \quad \text{for all } x \in [0,a]. \tag{6.68}$$

Integrating (6.68) over $[0,a]$ leads to the desired inequality (6.61). □

The next result ensures that the pair $(\underline{\mathcal{W}}_\lambda, \underline{b}_\lambda)$ is actually a lower solution of Problem (6.6) for λ large enough.

Lemma 6.6 *Let $u_0 \in X^h(b_0) \cap \mathbb{W}^{1,\infty}\big(0,b_0\big)$ and $(\underline{\mathcal{W}}_\lambda, \underline{b}_\lambda)$ defined by (6.54). There exists $\lambda \geqslant 1$ large enough such that $\underline{\mathcal{W}}_\lambda \leqslant u_0$ in $[0,b_0]$ and $\underline{b}_\lambda \leqslant b_0$. Then, $(\underline{\mathcal{W}}_\lambda, \underline{b}_\lambda)$ is a lower solution of Problem (6.6).*

Proof. According to (6.51), W_λ is a convex function. Thus, we have

$$W_\lambda(\eta) \leqslant \frac{h}{b_\lambda}(b_\lambda - \eta) \quad \text{for all } 0 \leqslant \eta \leqslant b_\lambda. \tag{6.69}$$

From the identity $u_0(\eta) = h + \displaystyle\int_0^\eta \frac{du_0}{d\eta}(s)ds$ for $0 \leqslant \eta \leqslant b_0$, we deduce that

$$u_0(\eta) \geqslant h - M\eta \quad \text{for all } 0 \leqslant \eta \leqslant b_0 \tag{6.70}$$

where $M = \|\dfrac{du_0}{d\eta}\|_{L^\infty(0,b_0)}$. From Lemma 6.5 (ii), $b_\lambda \to 0$ as $\lambda \to +\infty$. Then we can choose $\lambda \geqslant 1$ large enough so that

$$b_\lambda \leqslant \min\left(\frac{h}{M}, b_0\right). \tag{6.71}$$

Estimate (6.70) then becomes

$$u_0(\eta) \geqslant h - \frac{h}{b_\lambda}\eta \quad \text{for all } 0 \leqslant \eta \leqslant b_0$$

and we deduce from (6.69) that

$$u_0(\eta) \geqslant W_\lambda(\eta) \quad \text{for all } 0 \leqslant \eta \leqslant b_\lambda. \tag{6.72}$$

Defining $\underline{\mathcal{W}}_\lambda = W_\lambda$ and $\underline{b}_\lambda = b_\lambda$ as in (6.54), we deduce that the pair $(\underline{\mathcal{W}}_\lambda, \underline{b}_\lambda)$ is a lower solution for Problem (6.6). □

In view of Lemma 6.6 and the comparison principle Theorem 6.2, it follows that

$$\underline{b}_\lambda \leqslant b\big(\tau,(u_0,b_0)\big) \quad\text{and}\quad \underline{\mathcal{W}}_\lambda(\eta) \leqslant W\big(\eta,\tau,(u_0,b_0)\big) \text{ for all } \tau \geqslant 0, \eta \geqslant 0. \tag{6.73}$$

Next, we prove the monotonicity in time of the solution pair (W,b) of the time evolution Problem (6.6) with the two initial conditions $(\bar{\mathcal{W}},\bar{b})$ and $(\underline{\mathcal{W}}_\lambda,\underline{b}_\lambda)$. We recall that $(W(\eta,\tau,(u_0,b_0)),b(\tau,(u_0,b_0)))$ denotes the solution pair of Problem (6.6) with the initial conditions (u_0,b_0).

Lemma 6.7 *Let* $\big(\bar{\mathcal{W}},\bar{b}\big)$ *be the pair defined by* (6.22) *and* $(\underline{\mathcal{W}}_\lambda,\underline{b}_\lambda)$ *be the lower solution of Problem* (6.6) *defined by* (6.54).

(1) *The functions* $W\big(\eta,\tau,(\bar{\mathcal{W}},\bar{b})\big)$ *and* $b\big(\tau,(\bar{\mathcal{W}},\bar{b})\big)$ *are nonincreasing in time. Furthermore, there exist a positive constant* $\bar{b}_\infty$ *and a function* $\phi \in L^\infty(0,\bar{b}_\infty)$ *such that*

$$\lim_{\tau\to+\infty} W\big(\eta,\tau,(\bar{\mathcal{W}},\bar{b})\big) = \phi(\eta) \quad \textit{for all } \eta \in (0,\bar{b}_\infty), \tag{6.74}$$

$$\lim_{\tau\to+\infty} b\big(\tau,(\bar{\mathcal{W}},\bar{b})\big) = \bar{b}_\infty. \tag{6.75}$$

(2) *The function* $W\big(\eta,\tau,(\underline{\mathcal{W}}_\lambda,\underline{b}_\lambda)\big)$ *and* $b\big(\tau,(\underline{\mathcal{W}}_\lambda,\underline{b}_\lambda)\big)$ *are nondecreasing in time. Furthermore, there exist a positive constant* $\underline{b}_\infty$ *and a function* $\psi \in L^\infty(0,\underline{b}_\infty)$ *such that*

$$\lim_{\tau\to+\infty} W(\eta,\tau,(\underline{\mathcal{W}}_\lambda,\underline{b}_\lambda)) = \psi(\eta) \quad \textit{for all } \eta \in (0,\underline{b}_\infty), \tag{6.76}$$

$$\lim_{\tau\to+\infty} b\big(\tau,(\underline{\mathcal{W}}_\lambda,\underline{b}_\lambda)\big) = \underline{b}_\infty. \tag{6.77}$$

Proof. One can show that $W\big(\eta,\tau,(\bar{\mathcal{W}},\bar{b})\big)$ and $b\big(\tau,(\bar{\mathcal{W}},\bar{b})\big)$ are nonincreasing in time and that $W\big(\eta,\tau,(\underline{\mathcal{W}}_\lambda,\underline{b}_\lambda)\big)$ and $b\big(\tau,(\underline{\mathcal{W}}_\lambda,\underline{b}_\lambda)\big)$ are nondecreasing in time. Indeed, from (6.23) we have that

$$b\big(\tau,(u_0,b_0)\big) \leqslant \bar{b} \text{ and } W\big(\eta,\tau,(u_0,b_0)\big) \leqslant \bar{\mathcal{W}}(\eta) \text{ for all } \tau \geqslant 0 \text{ and } \eta \geqslant 0.$$

In particular, with $u_0 = \bar{\mathcal{W}}$ and $b_0 = \bar{b}$, we get

$$b\big(\tau,(\bar{\mathcal{W}},\bar{b})\big) \leqslant \bar{b} \text{ and } W\big(\eta,\tau,(\bar{\mathcal{W}},\bar{b})\big) \leqslant \bar{\mathcal{W}}(\eta) \quad \text{for all } \tau \geqslant 0 \text{ and } \eta \geqslant 0. \tag{6.78}$$

In view of (6.27), (6.41) and (6.44), we have that

$$0 \leqslant \bar{\mathcal{W}}(\eta). \tag{6.79}$$

Then, it follows from Proposition 3.3 that

$$0 \leqslant W\big(\eta,\tau,(\bar{\mathcal{W}},\bar{b})\big) \quad \text{for all } \tau \geqslant 0 \text{ and } \eta \geqslant 0. \tag{6.80}$$

Let $\sigma > 0$ be fixed. We apply Theorem 6.2 for (6.78) to obtain

$$b\big(\tau+\sigma,(\bar{\mathcal{W}},\bar{b})\big) \leqslant b\big(\sigma,(\bar{\mathcal{W}},\bar{b})\big) \text{ and } W\big(\eta,\tau+\sigma,(\bar{\mathcal{W}},\bar{b})\big) \leqslant W\big(\eta,\sigma,(\bar{\mathcal{W}},\bar{b})\big) \text{ for all } \tau \geqslant 0 \text{ and } \eta \geqslant 0.$$

Thus for each η, $W\big(\eta,\tau,(\bar{\mathcal{W}},\bar{b})\big)$ is nonincreasing in τ and from (6.80), it is bounded from below by zero. Therefore it has a limit ϕ as $\tau \to \infty$.

Also $b\big(\tau, (\bar{\mathcal{W}}, \bar{b})\big)$ is nonincreasing in τ and from (6.73) we deduce that it is bounded from below by $\underline{b}_\lambda$. Therefore it has a limit $\bar{b}_\infty$ as $\tau \to \infty$.

The same reasoning can be applied to prove that $W\big(\eta, \tau, (\underline{\mathcal{W}}_\lambda, \underline{b}_\lambda)\big)$ and $b\big(\tau, (\underline{\mathcal{W}}_\lambda, \underline{b}_\lambda)\big)$ are nondecreasing in time. Thus for each $\eta, W\big(\eta, \tau, (\underline{\mathcal{W}}_\lambda, \underline{b}_\lambda)\big)$ is nondecreasing in τ and it is bounded from above by the constant function h as follows from Proposition 3.3. Therefore it has a limit ψ as $\tau \to \infty$. Also, $b\big(\tau, (\underline{\mathcal{W}}_\lambda, \underline{b}_\lambda)\big)$ is nondecreasing in τ and bounded from above by $\bar{b}$ thanks to (6.23). Therefore it has a limit $\underline{b}_\infty$ as $\tau \to \infty$. □

Later we will show that ϕ and ψ coincide with the unique solution of Problem (2.14). To that purpose, we will derive in the Section 7 estimates for the free boundary Problem (6.6) in fixed domain.

7 A priori estimates for the solution of Problem (6.6) on the fixed domain

Definition 7.1 *We define*

$$\underline{b}(\tau) := b\big(\tau, (\underline{\mathcal{W}}_\lambda, \underline{b}_\lambda)\big) \text{ and } \underline{W}(\eta, \tau) := W\big(\eta, \tau, (\underline{\mathcal{W}}_\lambda, \underline{b}_\lambda)\big) \text{ for all } \tau > 0, 0 \leqslant \eta \leqslant \underline{b}(\tau),$$

$$\bar{b}(\tau) := b\big(\tau, (\bar{\mathcal{W}}, \bar{b})\big) \text{ and } \bar{W}(\eta, \tau) := W\big(\eta, \tau, (\bar{\mathcal{W}}, \bar{b})\big) \text{ for all } \tau > 0, 0 \leqslant \eta \leqslant \bar{b}(\tau).$$

We start by showing successive lemmas for the functions pair $(\underline{W}, \underline{b})$ and $(\bar{W}, \bar{b})$.

Lemma 7.2 *We have the following uniform bounds in time*

$$\underline{b}_\lambda \leqslant \underline{b}(\tau) \leqslant \bar{b}(\tau) \leqslant \bar{b} \quad \text{for all } \tau \geqslant 0 \tag{7.1}$$

and there exists a constant $\bar{h} \geqslant h$ such that

$$0 \leqslant \underline{W}(\eta, \tau) \leqslant \bar{W}(\eta, \tau) \leqslant \bar{h} \quad \text{for all } \tau \geqslant 0,\ 0 \leqslant \eta \leqslant \bar{b}. \tag{7.2}$$

Proof. It follows from (6.73) and (6.23) that

$$\underline{b}_\lambda \leqslant b\big(\tau, (u_0, b_0)\big) \leqslant \bar{b} \text{ for all } \tau \geqslant 0.$$

In particular, for $(u_0, b_0) = (\underline{\mathcal{W}}_\lambda, \underline{b}_\lambda)$, we obtain

$$\underline{b}_\lambda \leqslant \underline{b}(\tau) := b\big(\tau, (\underline{\mathcal{W}}_\lambda, \underline{b}_\lambda)\big) \leqslant \bar{b} \text{ for all } \tau \geqslant 0.$$

For $(u_0, b_0) = (\bar{\mathcal{W}}, \bar{b})$, we obtain

$$\underline{b}_\lambda \leqslant \bar{b}(\tau) := b\big(\tau, (\bar{\mathcal{W}}, \bar{b})\big) \leqslant \bar{b} \text{ for all } \tau \geqslant 0.$$

We know from (6.48), (6.54) and (6.79) that $0 \leqslant \underline{\mathcal{W}}_\lambda(\eta) \leqslant h \leqslant \bar{h}$ and $0 \leqslant \bar{\mathcal{W}}(\eta) \leqslant \bar{h}$ for all $\eta \in (0, \bar{b})$, which by Proposition 3.3 implies that

$$0 \leqslant \underline{W}(\eta, \tau) := W\big(\eta, \tau, (\underline{\mathcal{W}}_\lambda, \underline{b}_\lambda)\big) \leqslant \bar{h} \quad \text{for all } \tau \geqslant 0, 0 \leqslant \eta \leqslant \bar{b},$$

and

$$0 \leqslant \bar{W}(\eta, \tau) := W\big(\eta, \tau, (\bar{\mathcal{W}}, \bar{b}))\big) \leqslant \bar{h} \quad \text{for all } \tau \geqslant 0, 0 \leqslant \eta \leqslant \bar{b}.$$

We deduce from the comparison principle, Theorem 6.2, that (7.1) and (7.2) hold. □

Lemma 7.3 *For any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that*

$$\|\underline{b}\|_{C^1\left([\epsilon,\infty)\right)} \leqslant C_\epsilon, \quad \|\bar{b}\|_{C^1\left([\epsilon,\infty)\right)} \leqslant C_\epsilon.$$

Proof. By Lemmas 6.7 and 7.2, it follows that

$$\underline{b}'(\tau) \geqslant 0 \geqslant \bar{b}'(\tau), \quad \underline{b}_\lambda \leqslant \underline{b}(\tau) \leqslant \bar{b}(\tau) \leqslant \bar{b}, \quad 0 \leqslant \underline{W}(\eta,\tau) \leqslant \bar{W}(\eta,\tau) \leqslant \bar{h}.$$

Applying the differential equation for $\bar{b}(\tau)$ in (6.6) to $(\bar{W}, \bar{b})$, we immediately obtain

$$0 \geqslant \bar{b}'(\tau) = -\bar{W}_\eta\big(\bar{b}(\tau),\tau\big) - \frac{1}{2}\bar{b}(\tau) \geqslant -\frac{\bar{b}}{2},$$

since $\bar{W}_\eta\big(\bar{b}(\tau),\tau\big) \leqslant 0$. This yields the desired C^1 bound for the free boundary $\bar{b}(\tau)$. For the free boundary $\underline{b}(\tau)$, we apply a comparison argument. Let $\epsilon > 0$. We take $L = \dfrac{\underline{b}(\epsilon)}{2}$ and define

$$\widetilde{W}(\eta,\tau) := M\big(\underline{b}(\tau) - \eta\big) \ \text{ with } M > \frac{\bar{h}}{L} \ \text{ still to be fixed,}$$

with the constant $\bar{h} \geqslant h$ given in Lemma 7.2. Then we compare $\underline{W}$ and $\widetilde{W}$ on the region $\Omega := \{(\eta,\tau) : \underline{b}(\tau) - L < \eta < \underline{b}(\tau), \tau > \epsilon\}$, where $\widetilde{W}$ satisfies

$$\widetilde{W}_\tau - \widetilde{W}_{\eta\eta} - \frac{\eta}{2}\widetilde{W}_\eta = M\underline{b}'(\tau) + \frac{\eta}{2}M \geqslant 0$$

and

$$\widetilde{W}\big(\underline{b}(\tau),\tau\big) = 0 = \underline{W}\big(\underline{b}(\tau),\tau\big), \ \widetilde{W}\big(\underline{b}(\tau) - L,\tau\big) = ML > \bar{h} \geqslant \underline{W}\big(\underline{b}(\tau) - L,\tau\big).$$

Next, we choose $M > \dfrac{\bar{h}}{L}$ such that

$$\underline{W}_\eta(\eta,\epsilon) \geqslant -M \ \text{ for all } \ \eta \in \big[\underline{b}(\epsilon) - L, \underline{b}(\epsilon)\big],$$

which implies that

$$\underline{W}(\eta,\epsilon) = \underline{W}\big(\underline{b}(\epsilon),\epsilon\big) + \int_{\underline{b}(\epsilon)}^{\eta} \underline{W}_\eta\big(s,\epsilon\big)ds \leqslant M\big(\underline{b}(\epsilon) - \eta\big) = \widetilde{W}\big(\eta,\epsilon\big) \ \text{ for all } \ \eta \in \big[\underline{b}(\epsilon) - L, \underline{b}(\epsilon)\big].$$

Therefore we obtain, by the standard comparison principle, that

$$\widetilde{W} \geqslant \underline{W} \ \text{ in } \ \Omega.$$

Since $\widetilde{W}\big(\underline{b}(\tau),\tau\big) = \underline{W}\big(\underline{b}(\tau),\tau\big) = 0$, it follows that

$$-M = \widetilde{W}_\eta\big(\underline{b}(\tau),\tau\big) \leqslant \underline{W}_\eta\big(\underline{b}(\tau),\tau\big) \ \text{ for } \ \tau \geqslant \epsilon$$

and hence

$$0 \leqslant \underline{b}'(\tau) = -\underline{W}_\eta\big(\underline{b}(\tau),\tau\big) - \frac{1}{2}\underline{b}(\tau) \leqslant M - \frac{1}{2}\underline{b}_\lambda, \ \text{ for } \ \tau \geqslant \epsilon.$$

The desired estimate for $\underline{b}(\tau)$ thus follows. □

It will be necessary in the sequel to work on a fixed domain. To do so, we start by giving the transformation to the fixed domain $\hat{\Omega} := \big\{(y,\tau) \in (0,1) \times (0,\infty)\big\}$. We set

$$y_1 = \frac{\eta}{\underline{b}(\tau)}, \ \ \hat{W}^1(y_1,\tau) = \underline{W}(\eta,\tau) \ \text{ for all } \ \tau \geqslant 0, 0 \leqslant \eta \leqslant \underline{b}(\tau) \tag{7.3}$$

and

$$y_2 = \frac{\eta}{\bar{b}(\tau)}, \quad \hat{W}^2(y_2,\tau) = \bar{W}(\eta,\tau) \text{ for all } \tau \geqslant 0, 0 \leqslant \eta \leqslant \bar{b}(\tau). \tag{7.4}$$

The function $\hat{W}^1(y_1,\tau)$ satisfies the problem

$$\begin{cases} \hat{W}_\tau(y,\tau) = \dfrac{1}{\underline{b}^2(\tau)}\hat{W}_{yy}(y,\tau) + y\left(\dfrac{d\ln\big(\underline{b}(\tau)\big)}{d\tau} + \dfrac{1}{2}\right)\hat{W}_y(y,\tau), & \tau > 0,\ 0 < y < 1, \\ \hat{W}(0,\tau) = h, & \tau > 0, \\ \hat{W}(1,\tau) = 0, & \tau > 0, \\ \dfrac{1}{2}\dfrac{d\underline{b}^2(\tau)}{d\tau} + \dfrac{\underline{b}^2(\tau)}{2} = -\hat{W}_y(1,\tau), & \tau > 0, \\ \underline{b}(0) = b_0, & \\ \hat{W}(y,0) = u_0(b_0 y), & 0 \leqslant y \leqslant 1, \end{cases} \tag{7.5}$$

and the function $\hat{W}^2(y_2,\tau)$ satisfies the problem

$$\begin{cases} \hat{W}_\tau(y,\tau) = \dfrac{1}{\bar{b}^2(\tau)}\hat{W}_{yy}(y,\tau) + y\left(\dfrac{d\ln\big(\bar{b}(\tau)\big)}{d\tau} + \dfrac{1}{2}\right)\hat{W}_y(y,\tau), & \tau > 0,\ 0 < y < 1, \\ \hat{W}(0,\tau) = h, & \tau > 0, \\ \hat{W}(1,\tau) = 0, & \tau > 0, \\ \dfrac{1}{2}\dfrac{d\bar{b}^2(\tau)}{d\tau} + \dfrac{\bar{b}^2(\tau)}{2} = -\hat{W}_y(1,\tau), & \tau > 0, \\ \bar{b}(0) = b_0, & \\ \hat{W}(y,0) = u_0(b_0 y), & 0 \leqslant y \leqslant 1. \end{cases} \tag{7.6}$$

Let $r \geqslant 0$ and $\sigma > 0$. We define the domain

$$\Omega^r = (0,1) \times (r, r+\sigma). \tag{7.7}$$

Next, we define the extension of $\hat{W}^i$ on the domain $(-1,2) \times (0,\infty)$ as follows

$$\widetilde{W}^i(z,\tau) = \begin{cases} 2h - \hat{W}^i(-z,\tau) & \tau > 0, -1 < z < 0, \\ \hat{W}^i(z,\tau) & \tau > 0, 0 < z < 1, \\ -\hat{W}^i(2-z,\tau) & \tau > 0, 1 < z < 2. \end{cases} \tag{7.8}$$

In view of Problem (7.5), we deduce that $\widetilde{W}^1$ satisfies the following problem

$$\begin{cases} \widetilde{W}_\tau(z,\tau) = \dfrac{1}{\underline{b}^2(\tau)}\widetilde{W}_{zz}(z,\tau) + z\left(\dfrac{d\ln\big(\underline{b}(\tau)\big)}{d\tau} + \dfrac{1}{2}\right)\widetilde{W}_z(z,\tau), & \tau > 0,\ -1 < z < 1, \\ \widetilde{W}_\tau(z,\tau) = \dfrac{1}{\underline{b}^2(\tau)}\widetilde{W}_{zz}(z,\tau) + (z-2)\left(\dfrac{d\ln\big(\underline{b}(\tau)\big)}{d\tau} + \dfrac{1}{2}\right)\widetilde{W}_z(z,\tau), & \tau > 0,\ 1 < z < 2, \\ \widetilde{W}(-1,\tau) = 2h, & \tau > 0, \\ \widetilde{W}(2,\tau) = -h, & \tau > 0. \end{cases} \tag{7.9}$$

An analogous problem is also satisfied by $\widetilde{W}^2$ with $\bar{b}$ in place of $\underline{b}$ in (7.9).

The extensions $\widetilde{W^i}, i = 1, 2$, satisfy initial value problems of the form

$$\begin{cases} \widetilde{W}_\tau(z,\tau) = A(\tau)\widetilde{W}_{zz}(z,\tau) + B(z,\tau)\,\widetilde{W}_z(z,\tau), & \tau > 0,\ -1 < z < 2, \\ \widetilde{W}(-1,\tau) = 2h, & \tau > 0, \\ \widetilde{W}(2,\tau) = -h, & \tau > 0. \end{cases} \tag{7.10}$$

Theorem 7.4 *Let $p > 1$ and $r > \varepsilon > 0$. There exists a positive constant C which does not depend on r such that the solutions $\hat{W}^i$ of problems (7.5) and (7.6) satisfy the estimate*

$$\|\hat{W}^i\|_{W_p^{2,1}(\Omega^r)} \leqslant C. \tag{7.11}$$

Proof The coefficient A in (7.10), which only depends on τ, is positive and uniformly bounded away from zero on $[\varepsilon, \infty)$. Moreover, A satisfies $\|A\|_{C^1([\varepsilon,\infty))} \leqslant \mathcal{C}$ for some positive constant $\mathcal{C}$. The coefficient B in (7.10) is also uniformly bounded on $[-1, 2] \times [\varepsilon, \infty)$. The estimate (7.11) then follows from [12, Theorem 7.13], which gives interior estimates, and from its proof. This completes the proof of Theorem 7.4. □

Lemma 7.5 *We have*

$$W_p^{2,1}(\Omega^r) \subset C^{1+\alpha,\frac{1+\alpha}{2}}\big([0,1] \times [r, r+\sigma]\big) \quad \textit{with } \alpha = 1 - \frac{3}{p} \textit{ for all } p \in (3, \infty).$$

Proof Lemma 7.5 follows from Lemma 3.5 of [3, p.207]. □

Corollary 7.6 *There exists a positive constant C which does not depend on r such that the solution $\hat{W}^i$ of problems (7.5) and (7.6) satisfies the estimate*

$$\|\hat{W}^i(.,r)\|_{C^{1+\alpha}([0,1])} \leqslant C \textit{ for all } \alpha \in (0,1). \tag{7.12}$$

Proof We deduce from Lemma 7.5 that there exists some positives constants $\widetilde{C} > 0$ and $C > 0$ such that

$$\|\hat{W}^i\|_{C^{1+\alpha,\frac{1+\alpha}{2}}([0,1]\times[r,r+\sigma])} \leqslant \tilde{C}\|\hat{W}^i\|_{W_p^{2,1}(\Omega^r)} \leqslant C \text{ for all } \alpha \in (0,1), \tag{7.13}$$

which in turn implies that

$$\|\hat{W}^i(.,r)\|_{C^{1+\alpha}([0,1])} \leqslant C \text{ for all } \alpha \in (0,1). \tag{7.14}$$

This completes the proof of Corollary 7.6. □

8 Limit Problem as $\tau \to \infty$.

Theorem 8.1 *Let $(\psi, \underline{b}_\infty)$ be defined in Lemma 6.7. Then $(\psi, \underline{b}_\infty)$ is the unique stationary solution of Problem (2.14).*

Before proving this theorem, we need to show some preliminary results. Let $\hat{W}^1$ be defined as in (7.3). We also define

$$\hat{\psi}(y) = \psi(\eta), \quad y = \frac{\eta}{\underline{b}_\infty} \in [0,1] \ \text{ for } \ 0 \leqslant \eta \leqslant \underline{b}_\infty. \tag{8.1}$$

We start by showing the following result.

Lemma 8.2 *We have that* $\lim\limits_{r\to+\infty} ||\hat{W}^1(.,r) - \hat{\psi}||_{C^{1+\alpha}([0,1])} = 0$ *for all* $\alpha \in (0,1)$.

Proof. The proof of Lemma 8.2 follows from Corollary 7.6 and Lemma 6.7. □

Lemma 8.3 *We have that*

$$\hat{\psi}_{yy} \in L^p(0,1) \ \textit{for all} \ p > 1.$$

Proof. From Theorem 7.4, it follows that there exists a positive constant $C > 0$ such that

$$\left\|\hat{W}^i\right\|_{W_p^{2,1}\left(\Omega^r\right)} \leqslant C \ \text{ for all } \ p > 1.$$

which implies that

$$\int_r^{r+\sigma}\int_0^1 \left|\hat{W}^i_{yy}(y,s)\right|^p \mathrm{d}y \, \mathrm{d}s \leqslant C \tag{8.2}$$

With the change of variable $S = s - r$, the inequality (8.2) becomes

$$\int_0^{\sigma}\int_0^1 \left|\hat{W}^i_{yy}(y,S+r)\right|^p \mathrm{d}y \, \mathrm{d}S \leqslant C. \tag{8.3}$$

Thus, there exists $v \in L^p\left((0,1)\times(0,\sigma)\right)$ and a sequence $\{\hat{W}^{i,n}\}_{n\geqslant 0}$ of functions in $W_p^{2,1}\left((0,1)\times(0,\sigma)\right)$ such that

$$\hat{W}^{i,n}_{yy} \rightharpoonup v \text{ weakly in } L^p\left((0,1)\times(0,\sigma)\right) \quad \text{as } n \to +\infty \tag{8.4}$$

For all $\varphi \in \mathcal{D}\left((0,1)\times(0,\sigma)\right)$, we have that

$$\int_0^{\sigma}\int_0^1 \hat{W}^{i,n}_{yy}\varphi \, dyds \to \int_0^{\sigma}\int_0^1 v\varphi \, dyds \ \text{ as } n \to +\infty. \tag{8.5}$$

Integration by parts yields, in view of Theorem 7.4 and Lemma 8.2,

$$\int_0^{\sigma}\int_0^1 \hat{W}^{i,n}_{yy}\varphi \, dyds = \int_0^{\sigma}\int_0^1 \hat{W}^{i,n}\varphi_{yy} \, dyds \to \int_0^{\sigma}\int_0^1 \hat{\psi}\varphi_{yy} \, dyds \ \text{ as } n \to +\infty. \tag{8.6}$$

Now, since

$$\int_0^{\sigma}\int_0^1 \hat{\psi}\varphi_{yy} \, dyds = \left\langle \hat{\psi}_{yy}, \ \varphi \right\rangle_{\mathcal{D}',\mathcal{D}} \tag{8.7}$$

we deduce from (8.5), (8.6) and (8.7) that $v = \hat{\psi}_{yy} \in L^p(0,1)$, which completes the proof of Lemma 8.3. □

Next, we prove Theorem 8.1.

Proof of Theorem 8.1. The proof will be done through successive Lemmas. The first step of the proof consists in showing the following result.

Lemma 8.4 *We have* $\psi(0) = h$ *and* $\psi(\underline{b}_\infty) = 0$.

Proof. We start by showing that $\psi(0) = h$. Indeed, we have that (recall that $\underline{W}$ is nondecreasing in time)

$$\underline{\mathcal{W}}_\lambda(\eta) = \underline{W}(\eta, 0) \leqslant \underline{W}(\eta, \tau) \leqslant h. \tag{8.8}$$

Letting τ tend to $+\infty$, we deduce that

$$\underline{\mathcal{W}}_\lambda(\eta) \leqslant \psi(\eta) \leqslant h \quad \text{for all } \eta \in [0, \underline{b}_\infty].$$

Then, for $\eta = 0$, we obtain $\underline{\mathcal{W}}_\lambda(0) = h \leqslant \psi(0) \leqslant h$, that is $\psi(0) = h$.

Next, we prove that $\psi(\underline{b}_\infty) = 0$. We deduce from Lemma 8.2 that

$$\hat{W}^1(1, \tau) \to \hat{\psi}(1) \text{ as } \tau \to \infty, \tag{8.9}$$

which is equivalent to

$$\underline{W}\big(\underline{b}(\tau), \tau\big) \to \psi(\underline{b}_\infty) \text{ as } \tau \to \infty. \tag{8.10}$$

Since

$$\underline{W}\big(\underline{b}(\tau), \tau\big) = 0 \text{ for all } \tau > 0, \tag{8.11}$$

we deduce that indeed $\psi(\underline{b}_\infty) = 0$. □

The following result holds.

Lemma 8.5 *We have*

$$\frac{\underline{b}_\infty}{2} = -\psi_\eta(\underline{b}_\infty). \tag{8.12}$$

Proof. First, we prove the corresponding relation for $\hat{\psi}_y(1)$ and then we will conclude the result for ψ_η. We recall that

$$\frac{d\underline{b}(\tau)}{d\tau} + \frac{\underline{b}(\tau)}{2} = -\underline{W}_\eta(\underline{b}(\tau), \tau) \quad \text{for all } \tau > 0. \tag{8.13}$$

In view of the change of variables (7.3) for $\hat{W}^1$, the equation (8.13) becomes

$$\frac{1}{2}\frac{d\underline{b}^2(\tau)}{d\tau} + \frac{\underline{b}^2(\tau)}{2} = -\hat{W}^1_y(1, \tau) \text{ for all } \tau > 0. \tag{8.14}$$

Integrating (8.14) in time between τ and $\tau + \sigma$ and performing the change of variable $S = s - \tau$, we obtain

$$\frac{1}{2}\big(\underline{b}^2(\tau + \sigma) - \underline{b}^2(\tau)\big) + \frac{1}{2}\int_0^\sigma \underline{b}^2(S + \tau)\,\mathrm{d}S = -\int_0^\sigma \hat{W}^1_y(1, S + \tau)\,\mathrm{d}S. \tag{8.15}$$

Then, we deduce from Lemma 8.2 that

$$\hat{W}^1_y(1, S + \tau) \to \hat{\psi}_y(1) \text{ as } \tau \to +\infty \text{ in } C^\alpha\big([0, 1]\big) \text{ for all } \alpha \in (0, 1),$$

and recall that $\underline{b}(\tau) \to \underline{b}_\infty$ as $\tau \to +\infty$. Passing to the limit as $\tau \to +\infty$ in (8.15), we conclude that

$$\frac{\underline{b}_\infty^2}{2} = -\hat{\psi}_y(1). \tag{8.16}$$

Now, since $\psi_\eta(\eta) = \dfrac{1}{\underline{b}_\infty}\hat{\psi}_y(y),\ \ y = \dfrac{\eta}{\underline{b}_\infty}$ for all $0 \leqslant \eta \leqslant \underline{b}_\infty$ (see (8.1)), the relation (8.16) becomes

$$\frac{\underline{b}_\infty}{2} = -\psi_\eta(\underline{b}_\infty), \tag{8.17}$$

which completes the proof of Lemma 8.5. □

The last step of the proof of Theorem 8.1 consists in the following result.

Proposition 8.6 *The function $\psi \in C^\infty([0, \underline{b}_\infty])$ and satisfies the equation*

$$\psi_{\eta\eta} + \frac{\eta}{2}\psi_\eta = 0 \quad in\ (0, \underline{b}_\infty).$$

Before proving Proposition 8.6, we show the following lemma.

Lemma 8.7 *The function $\hat{\psi}$ satisfies*

$$\int_0^1 \hat{\psi}(y)\left(\frac{1}{\underline{b}_\infty^2}\ \varphi_{yy}(y) - \frac{y}{2}\ \varphi_y(y) - \frac{1}{2}\ \varphi(y)\right) \mathrm{d}y = 0 \tag{8.18}$$

for all test functions $\varphi \in \mathcal{D}(0,1)$.

Proof. Recall that the function $\hat{W}^1(y,\tau)$ satisfies Problem (7.5), in particular we have

$$\hat{W}^1_\tau(y,\tau) = \frac{1}{\underline{b}^2(\tau)}\hat{W}^1_{yy}(y,\tau) + y\left(\frac{d\ \ln\big(\underline{b}(\tau)\big)}{d\tau} + \frac{1}{2}\right)\hat{W}^1_y(y,\tau) \ \text{ for all }\ \tau > 0,\ 0 < y < 1. \tag{8.19}$$

From Problem (7.5), we have $\dfrac{1}{2}\dfrac{d\underline{b}^2(\tau)}{d\tau} + \dfrac{\underline{b}^2(\tau)}{2} = -\hat{W}^1_y(1,\tau)$ which implies that

$$\frac{d\underline{b}(\tau)}{d\tau} = \frac{-1}{\underline{b}(\tau)}\hat{W}^1_y(1,\tau) - \frac{\underline{b}(\tau)}{2}. \tag{8.20}$$

In view of the equality (8.20), the equation (8.19) becomes

$$\hat{W}^1_\tau(y,\tau) = \frac{1}{\underline{b}^2(\tau)}\hat{W}^1_{yy}(y,\tau) - \frac{y}{\underline{b}^2(\tau)}\ \hat{W}^1_y(1,\tau)\ \hat{W}^1_y(y,\tau) \ \text{ for all }\ \tau > 0,\ 0 < y < 1. \tag{8.21}$$

Next, we multiply (8.21) by the test function φ and integrate both sides of the equality on $(0,1) \times (\tau, \tau+\sigma)$ to obtain

$$\int_\tau^{\tau+\sigma}\int_0^1 \hat{W}^1_s(y,s)\varphi(y)\mathrm{d}y\ \mathrm{d}s = \int_\tau^{\tau+\sigma}\int_0^1 \left(\frac{1}{\underline{b}^2(s)}\hat{W}^1_{yy}(y,s) - \frac{y}{\underline{b}^2(s)}\ \hat{W}^1_y(1,s)\ \hat{W}^1_y(y,s)\right)\varphi(y)\ \mathrm{d}y\,\mathrm{d}s. \tag{8.22}$$

We integrate by parts the first term on the right-hand-side of (8.22) to obtain

$$\int_\tau^{\tau+\sigma}\int_0^1 \frac{1}{\underline{b}^2(s)}\hat{W}^1_{yy}(y,s)\varphi(y)\;\mathrm{d}y\,\mathrm{d}s = \int_\tau^{\tau+\sigma}\int_0^1 \frac{1}{\underline{b}^2(s)}\;\hat{W}^1(y,s)\;\varphi_{yy}(y)\;\mathrm{d}y\,\mathrm{d}s. \tag{8.23}$$

Next, we integrate by parts the second term on the right-hand-side of (8.22) to obtain

$$\int_\tau^{\tau+\sigma}\int_0^1 \frac{-y}{\underline{b}^2(s)}\;\hat{W}^1_y(1,s)\;\hat{W}^1_y(y,s)\;\varphi(y)\;\mathrm{d}y\,\mathrm{d}s =$$
$$\int_\tau^{\tau+\sigma}\int_0^1 \hat{W}^1(y,s)\Big(\frac{y}{\underline{b}^2(s)}\;\hat{W}^1_y(1,s)\;\varphi_y(y) + \frac{1}{\underline{b}^2(s)}\;\hat{W}^1_y(1,s)\;\varphi(y)\Big)\;\mathrm{d}y\,\mathrm{d}s. \tag{8.24}$$

We deduce from (8.23) and (8.24) that the right-hand-side of (8.22) becomes

$$\int_\tau^{\tau+\sigma}\int_0^1 \Big(\frac{1}{\underline{b}^2(s)}\hat{W}^1_{yy}(y,s) - \frac{y}{\underline{b}^2(s)}\;\hat{W}^1_y(1,s)\;\hat{W}^1_y(y,s)\Big)\,\varphi(y)\;\mathrm{d}y\,\mathrm{d}s =$$
$$\int_\tau^{\tau+\sigma}\int_0^1 \hat{W}^1(y,s)\Big(\frac{1}{\underline{b}^2(s)}\;\varphi_{yy}(y)+\frac{y}{\underline{b}^2(s)}\;\hat{W}^1_y(1,s)\;\varphi_y(y)+\frac{1}{\underline{b}^2(s)}\;\hat{W}^1_y(1,s)\;\varphi(y)\Big)\;\mathrm{d}y\,\mathrm{d}s. \tag{8.25}$$

Moreover, we have,

$$\int_\tau^{\tau+\sigma}\int_0^1 \hat{W}^1_s(y,s)\varphi(y)\mathrm{d}y\;\mathrm{d}s = \int_0^1 \hat{W}^1(y,\tau+\sigma)\varphi(y)\;\mathrm{d}y - \int_0^1 \hat{W}^1(y,\tau)\varphi(y)\,\mathrm{d}y. \tag{8.26}$$

We deduce from (8.22), (8.25) and (8.26) that

$$\int_0^1 \hat{W}^1(y,\tau+\sigma)\varphi(y)\;\mathrm{d}y - \int_0^1 \hat{W}^1(y,\tau)\varphi(y)\,\mathrm{d}y =$$
$$\int_\tau^{\tau+\sigma}\int_0^1 \hat{W}^1(y,s)\Big(\frac{1}{\underline{b}^2(s)}\;\varphi_{yy}(y)+\frac{y}{\underline{b}^2(s)}\;\hat{W}^1_y(1,s)\;\varphi_y(y)+\frac{1}{\underline{b}^2(s)}\;\hat{W}^1_y(1,s)\;\varphi(y)\Big)\;\mathrm{d}y\,\mathrm{d}s. \tag{8.27}$$

With the change of variables $S = s - \tau$, the equality (8.27) becomes

$$\int_0^1 \hat{W}^1(y,\tau+\sigma)\varphi(y)\;\mathrm{d}y - \int_0^1 \hat{W}^1(y,\tau)\varphi(y)\,\mathrm{d}y =$$
$$\int_0^\sigma\int_0^1 \hat{W}^1(y,S+\tau)\Big(\frac{1}{\underline{b}^2(S+\tau)}\;\varphi_{yy}(y)+\frac{y}{\underline{b}^2(S+\tau)}\;\hat{W}^1_y(1,S+\tau)\;\varphi_y(y)+\frac{1}{\underline{b}^2(S+\tau)}\;\hat{W}^1_y(1,S+\tau)\;\varphi(y)\Big)\;\mathrm{d}y\,\mathrm{d}S. \tag{8.28}$$

Furthermore, according to Lemma 8.2, we have

$$\lim_{\tau\to+\infty}\hat{W}^1(.,\tau) = \hat{\psi}\;\text{ in } C^\alpha\big([0,1]\big)\;\text{ for all }\;\alpha\in(0,1), \tag{8.29}$$

and

$$\hat{W}^1_y(1,\tau) \to \hat{\psi}_y(1)\;\text{ as }\;\tau\to+\infty, \tag{8.30}$$

and from (8.16), we have

$$\frac{\underline{b}^2_\infty}{2} = -\hat{\psi}_y(1). \tag{8.31}$$

According to Lemma 6.7, we recall that

$$\lim_{\tau\to+\infty} \underline{b}(\tau) = \underline{b}_\infty. \tag{8.32}$$

We recall that

$$0 \leqslant \hat{W}^1(y,\tau) \leqslant \bar{h} \ \text{ for all } \ \tau > 0, 0 < y < 1. \tag{8.33}$$

It follows that

$$\left|\hat{W}^1(y,\tau+\sigma)\varphi(y)\right| \leqslant \bar{h}\ \|\varphi\|_{L^\infty(0,1)}.$$

According to Lebesgue's Dominated Convergence Theorem,

$$\int_0^1 \hat{W}^1(y,\tau+\sigma)\varphi(y)\,\mathrm{d}y \to \int_0^1 \hat{\psi}(y)\varphi(y)\,\mathrm{d}y \quad \text{as } \tau\to\infty. \tag{8.34}$$

Similarly, we also have that

$$\int_0^1 \hat{W}^1(y,\tau)\varphi(y)\,\mathrm{d}y \to \int_0^1 \hat{\psi}(y)\varphi(y)\,\mathrm{d}y \quad \text{as } \tau\to\infty. \tag{8.35}$$

Now, we turn to the right-hand-side of (8.28). In view of (8.30), (8.31) and (8.32), we deduce that as $\tau\to\infty$

$$\int_0^\sigma\int_0^1 \hat{W}^1(y,S{+}\tau)\Big(\frac{1}{\underline{b}^2(S+\tau)}\ \varphi_{yy}(y){+}\frac{y}{\underline{b}^2(S+\tau)}\ \hat{W}^1_y(1,S{+}\tau)\ \varphi_y(y){+}\frac{1}{\underline{b}^2(S+\tau)}\ \hat{W}^1_y(1,S{+}\tau)\ \varphi(y)\Big)\,\mathrm{d}y\,\mathrm{d}S$$

$$\to \int_0^\sigma\int_0^1 \hat{\psi}(y)\Big(\frac{1}{\underline{b}^2_\infty}\ \varphi_{yy}(y) - \frac{y}{2}\ \varphi_y(y) - \frac{1}{2}\ \varphi(y)\Big)\,\mathrm{d}y\,\mathrm{d}S. \tag{8.36}$$

We conclude from (8.28) and (8.34)-(8.36) that

$$\int_0^1 \hat{\psi}(y)\Big(\frac{1}{\underline{b}^2_\infty}\ \varphi_{yy}(y) - \frac{y}{2}\ \varphi_y(y) - \frac{1}{2}\ \varphi(y)\Big)\,\mathrm{d}y = 0 \tag{8.37}$$

for all test functions $\varphi\in\mathcal{D}(0,1)$ which yields the result of Lemma 8.7. □

Finally, we present the proof of Proposition 8.6.

Proof of Proposition 8.6. From Lemma 8.3, we have that $\hat{\psi}_{yy}\in L^p(0,1)$. Then, by means of integration by parts, we obtain

$$\int_0^1 \hat{\psi}(y)\varphi_{yy}(y)\,\mathrm{d}y = \int_0^1 \hat{\psi}_{yy}(y)\varphi(y)\,\mathrm{d}y \tag{8.38}$$

and

$$\int_0^1 \hat{\psi}(y)\ \frac{y}{2}\ \varphi_y(y)\mathrm{d}y = -\int_0^1 \Big(\hat{\psi}_y(y)\ \frac{y}{2}\ \varphi(y) + \frac{1}{2}\hat{\psi}(y)\ \varphi(y)\Big)\,\mathrm{d}y \tag{8.39}$$

for all test function $\varphi\in\mathcal{D}(0,1)$. Hence, we deduce from (8.18) that

$$\int_0^1 \Big(\frac{1}{\underline{b}^2_\infty}\ \hat{\psi}_{yy}(y) + \frac{y}{2}\ \hat{\psi}_y(y)\Big)\,\varphi(y)\,\mathrm{d}y = 0, \tag{8.40}$$

for all $\varphi\in\mathcal{D}(0,1)$. In view of (8.1), we recall that

$$\hat{\psi}(y) = \psi(\eta), \quad y = \frac{\eta}{\underline{b}_\infty} \in [0,1] \ \text{ for } \ 0\leqslant\eta\leqslant\underline{b}_\infty. \tag{8.41}$$

This finally implies that

$$\psi \in C^{\infty}([0, \underline{b}_{\infty}]) \text{ and } \psi_{\eta\eta} + \frac{\eta}{2}\psi_{\eta} = 0 \text{ for all } 0 < \eta < \underline{b}_{\infty}. \tag{8.42}$$

This completes the proof of Proposition 8.6. □

We conclude that the pair $\left(\underline{W}(\eta,\tau) := W\big(\eta,\tau,(\underline{\mathcal{W}}_{\lambda},\underline{b}_{\lambda})\big), \underline{b}(\tau) := b\big(\tau,(\underline{\mathcal{W}}_{\lambda},\underline{b}_{\lambda})\big)\right)$ converges to $(\psi, \underline{b}_{\infty})$ as $\tau \to \infty$. Thanks to Lemma 8.4, Lemma 8.5 and Proposition 8.6, $(\psi, \underline{b}_{\infty})$ coincides with the unique stationary solution (U, a) of Problem (2.14). This completes the proof of Theorem 8.1. □

Similarly, one can show that $\left(W\big(\eta,\tau,(\bar{\mathcal{W}},\bar{b})\big), b\big(\tau,(\bar{\mathcal{W}},\bar{b})\big)\right)$ converges as $\tau \to \infty$ to $(\phi, \bar{b}_{\infty})$ which also coincides with the unique stationary solution (U, a) of Problem (2.14).

The main result of this article is main Theorem 2.1.

Recalling Lemmas 6.6 and 6.3, the main Theorem 2.1 implies the following result in the moving variables framework.

Theorem 8.8 *Let $u_0 \in X^h(b_0) \cap \mathbb{W}^{1,\infty}\big(0, b_0\big)$. Let $(W, b) = \big(W(\cdot,\cdot,(u_0,b_0)), b(\cdot,(u_0,b_0))\big)$ be the solution of Problem (6.6) with the initial data (u_0, b_0). Then*

$$\lim_{\tau \to +\infty} W(\eta,\tau) = U(\eta) \quad \textit{for all } \eta \in (0, a) \tag{8.43}$$

and

$$\lim_{\tau \to +\infty} b(\tau) = a \tag{8.44}$$

where (U, a) is the unique solution of the stationary Problem (2.14).

Proof. For all $\tau > 0$ and $\eta \geqslant 0$, we have that

$$W\big(\eta,\tau,(\underline{\mathcal{W}}_{\lambda},\underline{b}_{\lambda})\big) \leqslant W\big(\eta,\tau,(u_0,b_0)\big) \leqslant W\big(\eta,\tau,(\bar{\mathcal{W}},\bar{b})\big) \tag{8.45}$$

and

$$b\big(\tau,(\underline{\mathcal{W}}_{\lambda},\underline{b}_{\lambda})\big) \leqslant b\big(\tau,(u_0,b_0)\big) \leqslant b\big(\tau,(\bar{\mathcal{W}},\bar{b})\big). \tag{8.46}$$

According to Lemma 6.7 together with the fact that $(\psi, \underline{b}_{\infty}) = (\phi, \bar{b}_{\infty}) = (U, a)$, we deduce that

$$\lim_{\tau \to +\infty} W\big(\eta,\tau,(\bar{\mathcal{W}},\bar{b})\big) = \lim_{\tau \to +\infty} W(\eta,\tau,(\underline{\mathcal{W}}_{\lambda},\underline{b}_{\lambda})) = U(\eta), \tag{8.47}$$

$$\lim_{\tau \to +\infty} b\big(\tau,(\bar{\mathcal{W}},\bar{b})\big) = \lim_{\tau \to +\infty} b\big(\tau,(\underline{\mathcal{W}}_{\lambda},\underline{b}_{\lambda})\big) = a. \tag{8.48}$$

The result of Theorem 8.8 then follows from (8.45) and (8.46). □

This completes the proof of the main result of this article stated in Theorem 2.1 in Section 2.

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